Faculty of Science, Technology, Engineering and Mathematics
M208 Pure Mathematics

Additional exercises for Book B

Contents

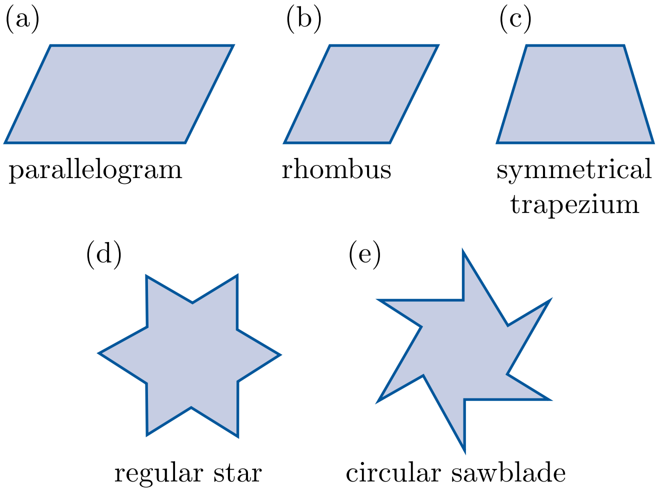
WEB 07700 3
2.1

Additional exercises for Unit B1

Section 1

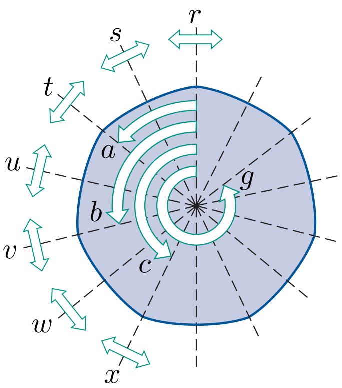
Additional Exercise B1

Describe geometrically the symmetries of each of the following figures.



Additional Exercise B2

The figure shown below is an equilateral curve heptagon (the shape of a 50p coin).



Its symmetries are:

- seven anticlockwise rotations: e, a, b, c, d, f and g about the centre through increasing multiples of $2\pi/7$
- seven reflections: r, s, t, u, v, w and x in the axes shown.

Determine the following composites:

$$b \circ d, \quad f \circ g, \quad c \circ v, \quad x \circ u.$$

Additional Exercise B3

Draw up a table of inverses for the symmetries of the equilateral curve heptagon, using the notation of Additional Exercise B2.

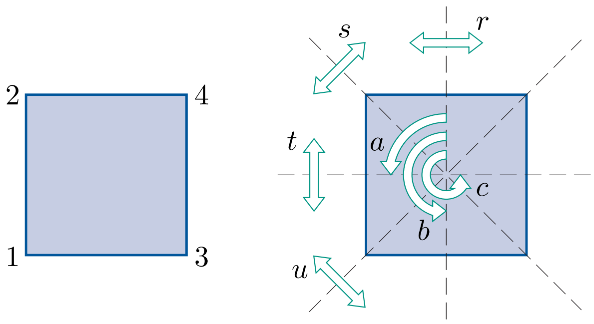
Additional Exercise B4

List the direct symmetries of the equilateral curve heptagon, using the notation of Additional Exercise B2. Show how each of the indirect symmetries can be obtained as a composite of the form $x \circ w$, where x is a direct symmetry and w is the indirect symmetry shown in Additional Exercise B2.

Section 2

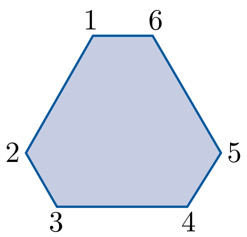
Additional Exercise B5

Write down the two-line symbol for each of the eight symmetries of the square, for the following labelling.



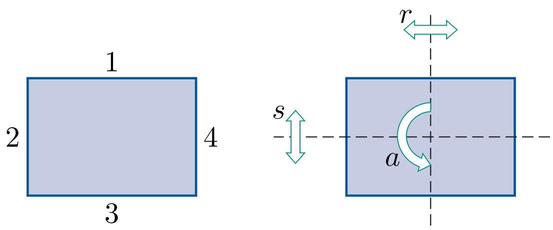
Additional Exercise B6

Describe geometrically the symmetries of the (non-regular) hexagon shown below. (The sides joining 1 to 6, 2 to 3 and 4 to 5 all have the same length, as do the sides joining 1 to 2, 3 to 4 and 5 to 6.) Write down the two-line symbol for each symmetry.



Additional Exercise B7

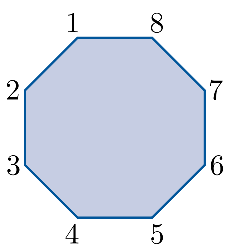
In the diagram below, the labels 1, 2, 3 and 4 refer to the locations of the four edges of the rectangle, rather than the locations of the vertices. (The position of the rectangle may be specified by the locations of the four edges instead of the four vertices. So our definition of two-line symbol still makes sense if we replace vertices by edges.)



For this labelling of the rectangle, write down the two-line symbol for each symmetry of the rectangle.

Additional Exercise B8

For the labelling of the regular octagon shown below, interpret geometrically each of the following two-line symbols.



(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 2 & 1 & 8 & 7 & 6 \end{pmatrix}$

Additional Exercise B9

Let

$$x = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Find the two-line symbol for each of the following composites:

$$x \circ x, \quad y \circ y, \quad x \circ y, \quad y \circ x.$$

Hence show that $(x \circ x) \circ x = e$, $(x \circ x) \circ y = y \circ x$ and $y \circ (x \circ x) = x \circ y$.

Section 3

Additional Exercise B10

Show that $(\mathbb{C}, +)$ is a group.

Additional Exercise B11

Let T be the set of integer multiples of 3; that is, $T = \{3k : k \in \mathbb{Z}\}$. Show that $(T, +)$ is a group.

Additional Exercise B12

Let $G = \{2^k : k \in \mathbb{Z}\}$. Show that $(G, \times)$ is a group.

Additional Exercise B13 Challenging

Show that $(\mathbb{R} - \{-1\}, \circ)$ is a group, where $\circ$ is defined by

$$x \circ y = x + y + xy.$$

(In Worked Exercise B11 you determined that $(\mathbb{R}, \circ)$, where $\circ$ is the binary operation above, is not a group. Be particularly careful when checking axiom G1, closure.)

Additional Exercise B14

Show that each of the following sets, with the binary operation given, is a group.

(a) $(\{1, 7, 9, 15\}, \times_{16})$ (b) $(\{0, 2, 4, 6, 8\}, +_{10})$

(c) $(\{1, 4, 9, 11, 16, 29\}, \times_{35})$

(d) $(\{3, 6, 9, 12\}, \times_{15})$

Additional Exercise B15

For each of the following, either show that the given set and binary operation form a group, or show that they do not.

(a) $(\mathbb{Z}^*, +)$

(b) $(\mathbb{R}^+, \div)$, where $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$

(c) $(\{4, 10, 16\}, \times_{18})$

(d) $(\{1, 4, 8\}, \times_{15})$

(e) $(T, \circ)$, where T is the subset $\{e, r, s, t, u\}$ of the set $S(\square)$ of symmetries of the square, and $\circ$ is function composition. (T contains the identity symmetry and all the indirect symmetries in $S(\square)$.)

Section 4

Additional Exercise B16

Given that the following tables are group tables, fill in the missing elements.

(a)

	e	a	b
e	e	a	b
a	a		
b	b		

(b)

	a	b	c	d
a	a	b	c	
b	b			a
c		d		
d	d			c

Additional Exercise B17

The following table is a group table.

	e	a	b	c	d	f	g	h
e	e	a	b	c	d	f	g	h
a	a	b	c	e	g	d	h	f
b	b	c	e	a	h	g	f	d
c	c	e	a	b	f	h	d	g
d	d	f	h	g	b	c	a	e
f	f	h	g	d	a	b	e	c
g	g	d	f	h	c	e	b	a
h	h	g	d	f	e	a	c	b

(a) Which element is the identity element?

(b) Write down the inverse of each of the elements $e, a, \ldots, h$.

(c) Is the group abelian?

Additional Exercise B18

Explain why each of the following Cayley tables is not a group table.

(a)

$\circ$	e	a	b	c
e	e	a	b	c
a	a	b	d	e
b	b	d	a	b
c	c	e	b	a

(b)

$\circ$	e	a	b	c
e	b	e	a	b
a	e	a	b	c
b	c	b	c	a
c	a	c	b	e

(c)

$\circ$	e	a	b	c	d
e	e	a	b	c	d
a	a	b	d	e	c
b	b	e	c	d	a
c	c	d	e	a	b
d	d	c	a	b	e

(d)

$\circ$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	f	b	c	d
b	b	d	a	e	f	c
c	c	f	e	d	b	a
d	d	b	c	f	a	e
f	f	c	d	a	e	b

Additional Exercise B19

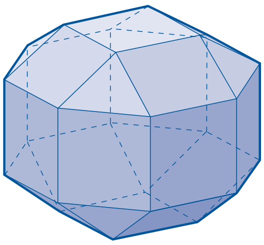
Let $(G, \circ)$ be a group with identity element e. Show that if G has an even number of elements, then there is an element $g \in G$ such that

$$g \circ g = e \quad \text{and} \quad g \neq e.$$

Section 5

Additional Exercise B20

The small rhombicuboctahedron, shown below, has 18 square faces and 8 faces that are equilateral triangles.



Use Strategy B2 to count the number of symmetries of the small rhombicuboctahedron, in two different ways, by considering the following types of faces in turn.

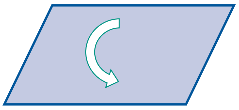
(a) Square faces of the type that have two edges joined to triangular faces and two edges joined to square faces.

(b) Triangular faces.

Solutions to additional exercises for Unit B1

Solution to Additional Exercise B1

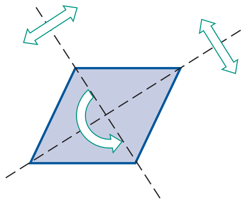
(a) Parallelogram



The symmetries are:

- the identity
- the rotation about the centre through π.

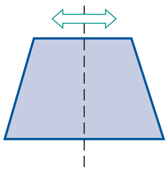
(b) Rhombus



The symmetries are:

- the identity
- the rotation about the centre through π
- the reflection in each diagonal.

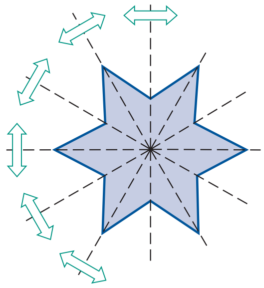
(c) Symmetrical trapezium



The symmetries are:

- the identity
- the reflection in the line bisecting the two parallel edges.

(d) Regular star

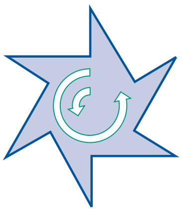


The symmetries are:

- the identity
- the five anticlockwise rotations about the centre through $\pi/3$, $2\pi/3$, π, $4\pi/3$ and $5\pi/3$
- six reflections, three in lines joining opposite vertices and three in lines joining opposite reflex angles.

(Only the reflections are shown on the figure.)

(e) Circular sawblade



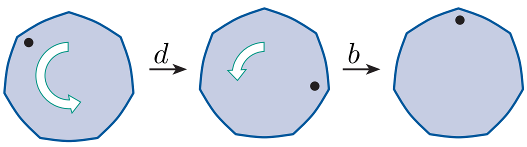
The symmetries are:

- the identity
- the five anticlockwise rotations about the centre through $\pi/3$, $2\pi/3$, π, $4\pi/3$ and $5\pi/3$.

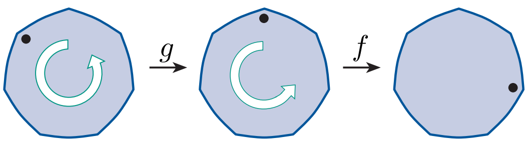
(Two of these rotations, through $2\pi/3$ and $5\pi/3$, are illustrated on the figure. The figure has no reflectional symmetries.)

Solution to Additional Exercise B2

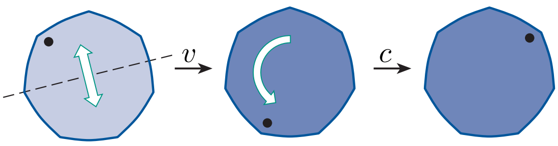
The required composites are as follows.



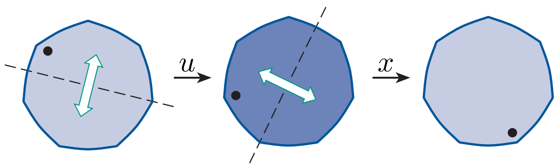
Hence $b \circ d = g$.



Hence $f \circ g = d$.



Hence $c \circ v = r$.



Hence $x \circ u = c$.

Solution to Additional Exercise B3

	Rotations							Reflections						
Element	e	a	b	c	d	f	g	r	s	t	u	v	w	x
Inverse	e	g	f	d	c	b	a	r	s	t	u	v	w	x

Note that each of the reflections is its own inverse.

Solution to Additional Exercise B4

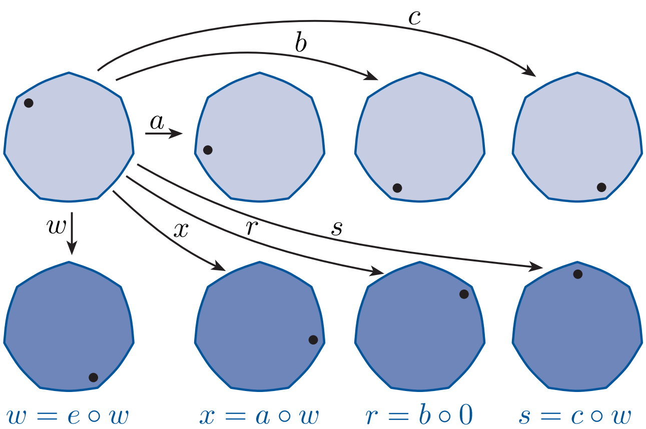
The direct symmetries are the identity and the six non-trivial rotations

$$a, b, c, d, f, g.$$

The indirect symmetries x, r and s are obtained by composing w with the rotations a, b and c, respectively:

$$x = a \circ w, \quad r = b \circ w, \quad s = c \circ w.$$

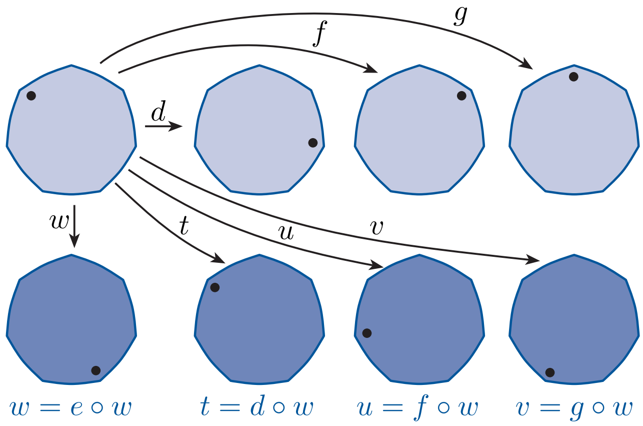
We can picture this as follows.



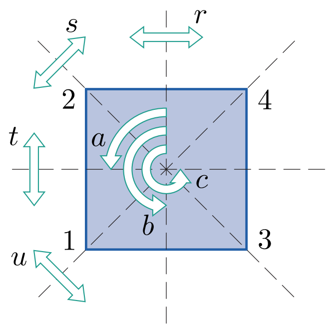
Similarly, the indirect symmetries t, u and v are obtained by composing w with the rotations d, f and g, respectively:

$$t = d \circ w, \quad u = f \circ w, \quad v = g \circ w.$$

We can picture this as follows.



Solution to Additional Exercise B5



The two-line symbols are

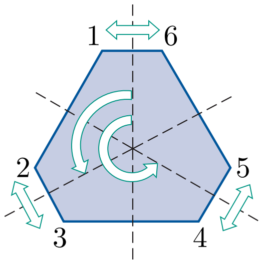
$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix},$$

$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix},$$

$$t = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}.$$

Solution to Additional Exercise B6



The symmetries are:

the identity,

$$\begin{pmatrix}1&2&3&4&5&6\\1&2&3&4&5&6\end{pmatrix};$$

rotation about the centre through $2\pi/3$ anticlockwise,

$$\begin{pmatrix}1&2&3&4&5&6\\3&4&5&6&1&2\end{pmatrix};$$

rotation about the centre through $4\pi/3$ anticlockwise,

$$\begin{pmatrix}1&2&3&4&5&6\\5&6&1&2&3&4\end{pmatrix};$$

reflection in the line bisecting the edges joining the locations $1, 6$ and $3, 4$,

$$\begin{pmatrix}1&2&3&4&5&6\\6&5&4&3&2&1\end{pmatrix};$$

reflection in the line bisecting the edges joining the locations $2, 3$ and $5, 6$,

$$\begin{pmatrix}1&2&3&4&5&6\\4&3&2&1&6&5\end{pmatrix};$$

reflection in the line bisecting the edges joining the locations $4, 5$ and $1, 2$,

$$\begin{pmatrix}1&2&3&4&5&6\\2&1&6&5&4&3\end{pmatrix}.$$

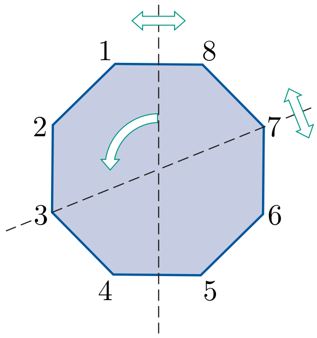
Solution to Additional Exercise B7

The two-line symbols are

$$e = \begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}, \quad a = \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix},$$

$$r = \begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}, \quad s = \begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}.$$

Solution to Additional Exercise B8



(a) $\begin{pmatrix}1&2&3&4&5&6&7&8\\3&4&5&6&7&8&1&2\end{pmatrix}$ represents anticlockwise rotation through $\pi/2$ about the centre.

(b) $\begin{pmatrix}1&2&3&4&5&6&7&8\\8&7&6&5&4&3&2&1\end{pmatrix}$ represents reflection in the line bisecting the edges joining locations $1, 8$ and $4, 5$.

(c) $\begin{pmatrix}1&2&3&4&5&6&7&8\\5&4&3&2&1&8&7&6\end{pmatrix}$ represents reflection in the line joining locations 3 and 7.

Solution to Additional Exercise B9

The two-line symbols are

$$x \circ x = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} \circ \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix},$$

$$\begin{aligned} y \circ y &= \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \circ \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \\ &= \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} = e, \end{aligned}$$

$$x \circ y = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} \circ \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix},$$

$$y \circ x = \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \circ \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}.$$

Using the results above, we obtain

$$\begin{aligned} (x \circ x) \circ x &= \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} \circ \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} \\ &= \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} = e, \end{aligned}$$

$$\begin{aligned} (x \circ x) \circ y &= \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} \circ \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \\ &= \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = y \circ x, \end{aligned}$$

$$y \circ (x \circ x) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = x \circ y.$$

Solution to Additional Exercise B10

The proof is similar to that in Worked Exercise B7.

We show that the four group axioms hold.

G1 For all $x, y \in \mathbb{C}$,

$$x + y \in \mathbb{C},$$

so $\mathbb{C}$ is closed under $+$.

G2 Addition of complex numbers is associative.

G3 We have $0 \in \mathbb{C}$, and for all $x \in \mathbb{C}$,

$$x + 0 = x = 0 + x,$$

so 0 is an identity element.

G4 For each $x \in \mathbb{C}$, we have $-x \in \mathbb{C}$, and

$$x + (-x) = 0 = (-x) + x,$$

so $-x$ is an inverse of x.

Hence $(\mathbb{C}, +)$ satisfies the four group axioms, and so is a group.

Solution to Additional Exercise B11

We show that the four group axioms hold.

G1 Let $m, n \in T$. Then $m = 3k$ and $n = 3l$ for some $k, l \in \mathbb{Z}$. Hence

$$m + n = 3k + 3l = 3(k + l),$$

which is an element of T, since $k + l \in \mathbb{Z}$. Thus T is closed under $+$.

G2 Addition of integers is associative.

G3 We have $0 \in T$ (since $0 = 3 \times 0$), and for all $n \in T$,

$$n + 0 = n = 0 + n,$$

so 0 is an identity element.

G4 Let $m \in T$. Then $m = 3k$ for some $k \in \mathbb{Z}$. Hence $-m = -3k = 3(-k) \in T$, and we have

$$m + (-m) = 0 = (-m) + m,$$

so $-m$ is an inverse of m in T. Thus each element of T has an inverse in T with respect to addition.

Hence $(T, +)$ satisfies the four group axioms, and so is a group.

Solution to Additional Exercise B12

We show that the four group axioms hold.

G1 Let $x, y \in G$. Then $x = 2^k$ and $y = 2^l$ for some $k, l \in \mathbb{Z}$. Hence

$$x \times y = 2^k \times 2^l = 2^{k+l} \in G.$$

Thus G is closed under multiplication.

G2 Multiplication of numbers is associative.

G3 We have $1 \in G$, since $1 = 2^0$, and for all $x \in G$,

$$x \times 1 = x = 1 \times x,$$

so 1 is an identity element.

G4 Let $x \in G$. Then $x = 2^k$ for some $k \in \mathbb{Z}$. Now $1/x = 1/2^k = 2^{-k} \in G$, and

$$x \times \frac{1}{x} = 1 = \frac{1}{x} \times x,$$

so $1/x$ is an inverse of x. Thus each element of G has an inverse in G with respect to multiplication.

Hence $(G, \times)$ satisfies the four group axioms, and so is a group.

Solution to Additional Exercise B13

We show that $(\mathbb{R} - \{-1\}, \circ)$ satisfies the group axioms.

G1 Let $x, y \in \mathbb{R} - \{-1\}$. Then

$$x \circ y = x + y + xy \in \mathbb{R}.$$

To show that $x + y + xy \in \mathbb{R} - \{-1\}$, we also need to show that $x + y + xy \neq -1$. To do that, we can try a proof by contradiction. So suppose that

$$x + y + xy = -1.$$

Then

$$1 + x + y + xy = 0,$$

which gives

$$1 + x + y(1 + x) = 0;$$

that is,

$$(1 + y)(1 + x) = 0.$$

So $x = -1$ or $y = -1$. However, this is a contradiction, since $x, y \in \mathbb{R} - \{-1\}$. Hence we can conclude that $x + y + xy \neq -1$. This completes the proof that $\mathbb{R} - \{-1\}$ is closed under $\circ$.

G2 In Worked Exercise B11 we showed that $\circ$ is associative for all elements of $\mathbb{R}$, and hence it is associative for all elements of $\mathbb{R} - \{-1\}$.

G3 In Worked Exercise B11 we showed that 0 is an identity element for $\circ$ on $\mathbb{R}$. Since $0 \in \mathbb{R} - \{-1\}$, it follows that 0 is an identity element for $\circ$ on $\mathbb{R} - \{-1\}$.

G4 Let $x \in \mathbb{R} - \{-1\}$. In Worked Exercise B11 we showed that if y is an element of $\mathbb{R}$ such that $x \circ y = 0$, then $y = -x/(1+x)$.

The manipulation of equations that we used to show this is valid when reversed, so it follows that if $y = -x/(1+x)$, then $x \circ y = 0$, and hence also that $y \circ x = 0$, since $x \circ y = y \circ x$. (Alternatively, you can check explicitly that $x \circ (-x/(1+x)) = 0$ and $(-x/(1+x)) \circ x = 0$.)

Also, $y = -x/(1+x) \neq -1$, since $x \neq 1 + x$, so $y = -x/(1+x) \in \mathbb{R} - \{-1\}$.

So $y = -x/(1+x)$ is an inverse of x in $\mathbb{R} - \{-1\}$.

Thus the four group axioms are satisfied, and hence $(\mathbb{R} - \{-1\}, \circ)$ is a group.

Solution to Additional Exercise B14

(a) A Cayley table for $(\{1, 7, 9, 15\}, \times_{16})$ is as follows.

$\times_{16}$	1	7	9	15
1	1	7	9	15
7	7	1	15	9
9	9	15	1	7
15	15	9	7	1

G1 Every element in the body of the table is in $\{1, 7, 9, 15\}$, so this set is closed under $\times_{16}$.
G2 The operation $\times_{16}$ is associative.
G3 From the table we see that 1 is an identity.
G4 From the table, we see that each element in $\{1, 7, 9, 15\}$ is self-inverse, so this set contains an inverse of each element.

Hence $(\{1, 7, 9, 15\}, \times_8)$ satisfies the four group axioms, and so is a group.

(b) A Cayley table for $(\{0, 2, 4, 6, 8\}, +_{10})$ is as follows.

$+_{10}$	0	2	4	6	8
0	0	2	4	6	8
2	2	4	6	8	0
4	4	6	8	0	2
6	6	8	0	2	4
8	8	0	2	4	6

G1 Every element in the body of the table is in $\{0, 2, 4, 6, 8\}$, so this set is closed under $+_{10}$.
G2 The operation $+_{10}$ is associative.
G3 From the table, we see that 0 is an identity element.
G4 From the table, we see that each element has an inverse, as follows.

Element	0	2	4	6	8
Inverse	0	8	6	4	2

Hence $(\{0, 2, 4, 6, 8\}, +_{10})$ satisfies the four group axioms, and so is a group.

(c) A Cayley table for $(\{1, 4, 9, 11, 16, 29\}, \times_{35})$ is as follows.

$\times_{35}$	1	4	9	11	16	29
1	1	4	9	11	16	29
4	4	16	1	9	29	11
9	9	1	11	29	4	16
11	11	9	29	16	1	4
16	16	29	4	1	11	9
29	29	11	16	4	9	1

G1 Every element in the body of the table is in $\{1, 4, 9, 11, 16, 29\}$, so this set is closed under the operation $\times_{35}$.
G2 The operation $\times_{35}$ is associative.
G3 From the table, we see that 1 is an identity.
G4 From the table, we see that each element has an inverse, as follows.

Element	1	4	9	11	16	29
Inverse	1	9	4	16	11	29

Hence $(\{1, 4, 9, 11, 16, 29\}, \times_{15})$ satisfies the four group axioms, and so is a group.

(d) A Cayley table for $(\{3, 6, 9, 12\}, \times_{15})$ is as follows.

$\times_{15}$	3	6	9	12
3	9	3	12	6
6	3	6	9	12
9	12	9	6	3
12	6	12	3	9

G1 Every element in the body of the table is in $\{3, 6, 9, 12\}$, so this set is closed under $\times_{15}$.
G2 The operation $\times_{15}$ is associative.
G3 From the table, we see that 6 is an identity.
G4 From the table, we see that each element has an inverse, as follows.

Element	3	6	9	12
Inverse	12	6	9	3

Hence $(\{3,6,9,12\}, \times_{15})$ satisfies the four group axioms, and so is a group.

Solution to Additional Exercise B15

(a) Axiom G1 fails: $\mathbb{Z}^*$ is not closed under addition. For example, $1, -1 \in \mathbb{Z}^*$ but $1 + (-1) = 0 \notin \mathbb{Z}^*$.

Hence $(\mathbb{Z}^*, +)$ is not a group.

(b) The set $\mathbb{R}^+$ is closed under division, since dividing a positive real number by a positive real number gives a positive real number.

However, axiom G2 fails: division is not associative on $\mathbb{R}^+$. For example,

$$(8 \div 4) \div 2 = 2 \div 2 = 1,$$

whereas

$$8 \div (4 \div 2) = 8 \div 2 = 4.$$

Hence $(\mathbb{R}^+, \div)$ is not a group.

(c) A Cayley table for $(\{4,10,16\}, \times_{18})$ is as follows.

$\times_{18}$	4	10	16
4	16	4	10
10	4	10	16
16	10	16	4

G1 Every element in the body of the table is in $\{4,10,16\}$, so this set is closed under $\times_{18}$.
G2 The operation $\times_{18}$ is associative.
G3 From the table, we see that 10 is an identity.
G4 From the table, we see that each element has an inverse, as follows.

Element	4	10	16
Inverse	16	10	4

Hence $(\{4,10,16\}, \times_{18})$ satisfies the four group axioms, and so is a group.

(d) A Cayley table for $(\{1,4,8\}, \times_{15})$ is as follows.

$\times_{15}$	1	4	8
1	1	4	8
4	4	1	2
8	8	2	4

The table contains the integer 2, which is not in the set $\{1,4,8\}$, so this set is not closed under $\times_{15}$. That is, axiom G1 fails.

Hence $(\{1,4,8\}, \times_{15})$ is not a group.

(e) The set T consists of all the indirect symmetries of the square (that is, the reflections), together with the identity symmetry. Since we know that the composition of two indirect symmetries is always a direct symmetry, T is not closed under function composition; for example, $r \circ s = c$ and $c \notin T$. Therefore Axiom G1 fails, so $(T, \circ)$ is not a group.

Note that the other three axioms all hold: function composition is associative (G2), T contains the identity symmetry e (G3), and each of the elements of T is self-inverse and so has an inverse in T (G4).

Solution to Additional Exercise B16

We use the property that in a group table each element appears once in each row and once in each column.

(a)

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

(b)

	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Solution to Additional Exercise B17

(a) The first row and the first column repeat the borders of the table, so the identity is e.

(b) The table of inverses is as follows.

Element	e	a	b	c	d	f	g	h
Inverse	e	c	b	a	h	g	f	d

(c) The group is non-abelian because the table is not symmetrical with respect to the main diagonal; for example, $h \circ g = c$, whereas $g \circ h = a$.

Solution to Additional Exercise B18

(a) The set is not closed under $\circ$: a new element d occurs in the body of the table, so axiom G1 fails.

(b) Here the set is closed under $\circ$, and we see from the row labelled a and column labelled a that a is an identity element.

However, a does not occur symmetrically with respect to the main diagonal, so this is not a group. (The fact that a does not occur symmetrically tells us that some of the elements do not have inverses. For example, $b \circ c = a$ but $c \circ b = b \neq a$.)

Alternatively, b occurs twice in the row labelled e (and also in the column labelled b), so this is not a group.

(c) Again, the set is closed under $\circ$, and we see from the row labelled e and column labelled e that e is an identity element.

However, e does not appear symmetrically with respect to the main diagonal, so this is not a group.

Alternatively, the operation $\circ$ is not associative because

$$a \circ (b \circ d) = a \circ a = b$$

but

$$(a \circ b) \circ d = d \circ d = e.$$

Hence

$$a \circ (b \circ d) \neq (a \circ b) \circ d,$$

so axiom G2 fails.

(d) Here the only axiom that fails is G2 (associativity). For example,

$$a \circ (b \circ d) = a \circ f = d$$

but

$$(a \circ b) \circ d = f \circ d = e.$$

Hence

$$a \circ (b \circ d) \neq (a \circ b) \circ d,$$

so axiom G2 fails.

Solution to Additional Exercise B19

Let G be a group such that $|G|$ is even. We know that, for each element $g \in G$, either g is self-inverse or g and g^{-1} are distinct elements that are inverses of each other.

It follows that the number of elements that are self-inverse must be even.

However, e is self-inverse, so there must be at least one element $g \in G$ such that

$$g \circ g = e \quad \text{and} \quad g \neq e.$$

Solution to Additional Exercise B20

(a) Consider the square faces of the type described, with two edges joined to triangular faces and two edges joined to square faces.

1. The small rhombicuboctahedron has 12 square faces of this type.
2. Only 4 of the 8 symmetries of a square face of this type give symmetries of the rhombicuboctahedron. (These four symmetries are the identity symmetry, the rotation through π, and the two reflections in axes of symmetry parallel to the sides of the square.)
3. Hence, by Strategy B2, the number of symmetries of the rhombicuboctahedron is $12 \times 4 = 48$.

(b) Now consider the triangular faces, which are equilateral triangles.

1. The small rhombicuboctahedron has 8 triangular faces.
2. All 6 symmetries of a triangular face give symmetries of the rhombicuboctahedron.
3. Hence, by Strategy B2, the number of symmetries of the rhombicuboctahedron is $8 \times 6 = 48$.

(As expected, we obtain the same number of symmetries whichever type of face we consider.)

(The symmetries of a small rhombicuboctahedron were also counted in Subsection 5.2 of Unit B1, using the square faces of the other type: those whose four edges are all joined to square faces.)

Additional exercises for Unit B2

Section 1

Additional Exercise B21

Prove the following statements using the three subgroup properties.

(a) $(A, \times)$ is a subgroup of $(\mathbb{R}^*, \times)$, where $A = \{10^k : k \in \mathbb{Z}\}$.

(b) $(B, +)$ is a subgroup of $(\mathbb{C}, +)$, where $B = \{z \in \mathbb{C} : z = x + ix,\ x \in \mathbb{R}, \}$.

(c) $(C, +_{15})$ is a subgroup of $(\mathbb{Z}_{15}, +_{15})$, where $C = \{0, 3, 6, 9, 12\}$.

Additional Exercise B22

Let X be the subset of $\mathbb{R}^2$ consisting of all the points not on the y-axis; that is,

$$X = \{(a, b) \in \mathbb{R}^2 : a \neq 0\},$$

and $*$ be the binary operation on X defined by

$$(a, b) * (c, d) = (ac, ad + b).$$

You saw in Worked Exercise B18 that $(X, *)$ is a group, in which the identity is $(1, 0)$ and the inverse of each element (a, b) of X is given by $(1/a, -b/a)$.

(a) Let

$$C = \{(a, b) \in X : b = 0\}.$$

Show that $(C, *)$ is a subgroup of $(X, *)$.

(b) Let

$$D = \{(a, b) \in X : a > 0\}.$$

Show that $(D, *)$ is a subgroup of $(X, *)$.

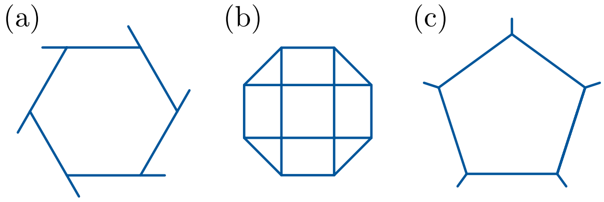
(c) Let

$$E = \{(a, b) \in X : a = b\}.$$

Show that $(E, *)$ is not a subgroup of $(X, *)$.

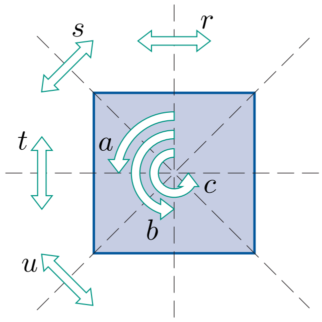
Additional Exercise B23

Describe geometrically all the symmetries of each of the following modified regular polygons.

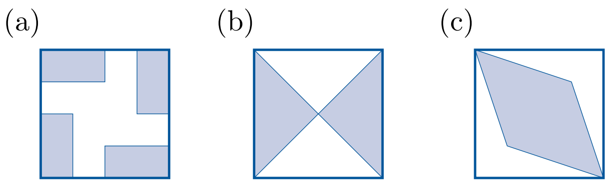


Additional Exercise B24

The non-identity elements of $S(\square)$ are shown below.

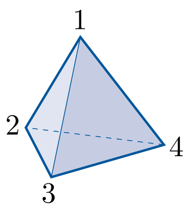


Find three subgroups of $S(\square)$ by listing the elements of the symmetry groups of each of the following modified squares.



Additional Exercise B25

Write down, as two-line symbols, the elements of the subgroup of $S(\text{tet})$ that consists of the symmetries of a tetrahedron that fix the edge joining the vertices at locations 1 and 3. Remember that an edge can be fixed without each point on the edge being fixed.



Section 2

Additional Exercise B26

Write down the elements of the group $(U_{16}, \times_{16})$, and use any method to find the order of every element of this group.

Section 3

Additional Exercise B27

Write down the subgroup generated by each element of the group $(\mathbb{Z}_{14}, +_{14})$, and state the order of each element of this group.

Additional Exercise B28

State the order of each element of the group $S(\text{⬠})$, and describe the subgroup that it generates.

Additional Exercise B29

Determine which of the following groups are cyclic, and state all the generators of those that are cyclic.

(a) $(U_{15}, \times_{15})$ (b) $(\mathbb{Z}_7^*, \times_7)$ (c) $(\mathbb{Z}_{11}^*, \times_{11})$

(d) $(U_{14}, \times_{14})$

Additional Exercise B30

Write down the order of each of the following elements of $S(\bigcirc)$.

(a) $r_{2\pi/9}$ (b) $r_{3\pi/7}$ (c) r_3

Additional Exercise B31

Find the order of each of the following elements in the given group.

(a) 3 in $(\mathbb{Z}_{15}, +_{15})$ (b) 9 in $(\mathbb{Z}_{21}, +_{21})$

Additional Exercise B32

Write down all the generators of each of the following groups.

(a) $(\mathbb{Z}_{14}, +_{14})$ (b) $(\mathbb{Z}_{16}, +_{16})$

Additional Exercise B33

Find all the cyclic subgroups of the group $(U_{24}, \times_{24})$.

Additional Exercise B34

Show that $(\mathbb{Z}_{13}^*, \times_{13})$ is a cyclic group, and hence find all its subgroups.

Section 4

Additional Exercise B35

Find an isomorphism from the first group to the second group in each of the following cases.

(a) $(U_{12}, \times_{12})$ and $(U_8, \times_8)$

(b) $(\mathbb{Z}_{11}^*, \times_{11})$ and $(\mathbb{Z}_{10}, +_{10})$

(c) $(U_9, \times_9)$ and $(U_{14}, \times_{14})$

The solutions to the following exercises may be helpful: Exercise B73 (for $(U_{12}, \times_{12})$), Additional Exercise B29 (for $(\mathbb{Z}_{11}^*, \times_{11})$ and $(U_{14}, \times_{14})$), Exercise B79 (for $(U_9, \times_9)$. Also, $(U_8, \times_8)$ is discussed in Subsection 4.1 of Unit B2.)

Additional Exercise B36

Determine whether the groups $(U_{16}, \times_{16})$ and $(U_{24}, \times_{24})$ are isomorphic.

The solutions to Additional Exercises B26 and B33 may be helpful.

Additional Exercise B37 Challenging

Additional Exercise B35 asked you to find an isomorphism from $(U_{12}, \times_{12})$ to $(U_8, \times_8)$. How many such isomorphisms are there?

Additional Exercise B38 Challenging

Use the formal definition of isomorphic groups to prove Theorem B44; that is, the relation 'is isomorphic to' is an equivalence relation on the collection of all groups.

Solutions to additional exercises for Unit B2

Solution to Additional Exercise B21

(a) We have $A \subseteq \mathbb{R}^*$, and the binary operation $\times$ is the same on each set.

We show that the three subgroup properties hold.

SG1 Let $x, y \in A$; then $x = 10^m$ and $y = 10^n$, for some $m, n \in \mathbb{Z}$. So

$$xy = 10^m \times 10^n = 10^{m+n}.$$

Since $m + n$ is an integer, this shows that $xy \in A$. Thus A is closed under multiplication.

SG2 The identity in $(\mathbb{R}^*, \times)$ is 1, and $1 = 10^0$, so $1 \in A$. Thus A contains the identity.

SG3 Let $x \in A$; then $x = 10^m$ for some $m \in \mathbb{Z}$. The inverse of x in $(\mathbb{R}^*, \times)$ is

$$\frac{1}{x} = \frac{1}{10^m} = 10^{-m}.$$

Since $-m$ is an integer, this shows that the inverse of x is in A. Thus A contains the inverse of each of its elements.

Hence $(A, \times)$ satisfies the three subgroup properties, and so is a subgroup of $(\mathbb{R}^*, \times)$.

(b) We have $B \subseteq \mathbb{C}$, and the binary operation $+$ is the same on each set.

We show that the three subgroup properties hold.

SG1 Let $z_1, z_2 \in B$; then $z_1 = x_1 + ix_1$ and $z_2 = x_2 + ix_2$ for some $x_1, x_2 \in \mathbb{R}$. So

$$\begin{aligned} z_1 + z_2 &= (x_1 + ix_1) + (x_2 + ix_2) \\ &= (x_1 + x_2) + i(x_1 + x_2). \end{aligned}$$

This complex number is in B, since the real and imaginary parts are equal. Hence B is closed under addition.

SG2 The identity in $(\mathbb{C}, +)$ is

$$0 = 0 + i0,$$

which is in B, since the real and imaginary parts are equal.

SG3 Let $z \in B$. Then $z = x + ix$ for some $x \in \mathbb{R}$. The inverse of z in $(\mathbb{C}, +)$ is

$$\begin{aligned} -z &= -(x + ix) \\ &= -x + i(-x), \end{aligned}$$

which is in B, since the real and imaginary parts are equal. Thus B contains the inverse of each of its elements.

Hence $(B, +)$ satisfies the three subgroup properties, and so is a subgroup of $(\mathbb{C}, +)$.

(c) We have $C \subseteq \mathbb{Z}_{15}$, and the binary operation $+_{15}$ is the same on each set.

We show that the three subgroup properties hold.

For this finite set, we use a Cayley table.

$+_{15}$	0	3	6	9	12
0	0	3	6	9	12
3	3	6	9	12	0
6	6	9	12	0	3
9	9	12	0	3	6
12	12	0	3	6	9

SG1 Every element in the body of the table is in C, so C is closed under $+_{15}$.

SG2 The identity in $(\mathbb{Z}_{15}, +_{15})$ is 0, which is in C.

SG3 From the table, we see that each element has an inverse in C.

Element	0	3	6	9	12
Inverse	0	12	9	6	3

Hence $(C, +_{15})$ satisfies the three subgroup properties, and so is a subgroup of $(\mathbb{Z}_{15}, +_{15})$.

Solution to Additional Exercise B22

(a) We have

$$\begin{aligned} C &= \{(a, b) \in X : b = 0\} \\ &= \{(a, 0) : a \in \mathbb{R}, a \neq 0\}. \end{aligned}$$

Now $C \subseteq X$, and the operation $*$ is the same on both sets. We show that $(C, *)$ satisfies the three subgroup properties.

SG1 Let $(a, 0), (c, 0) \in C$; then $a \neq 0$ and $c \neq 0$. We have

$$(a, 0) * (c, 0) = (ac, a \times 0 + 0) = (ac, 0).$$

This point is in C because the first coordinate is non-zero (since $a \neq 0$ and $c \neq 0$) and the second coordinate is zero.

Thus C is closed under $*$.

SG2 The identity in X is $(1, 0)$, and $(1, 0)$ belongs to C because the first coordinate is non-zero and the second coordinate is zero.

SG3 Let $(a, 0) \in C$. Then $a \neq 0$, and

$$(a, 0)^{-1} = (1/a, -0/a) = (1/a, 0).$$

This point belongs to C because the first coordinate is non-zero and the second coordinate is zero. Thus C contains the inverse of each of its elements.

Hence $(C, *)$ satisfies the three subgroup properties, so $(C, *)$ is a subgroup of $(X, *)$.

(b) We have

$$\begin{aligned} D &= \{(a, b) \in X : a > 0\} \\ &= \{(a, b) : a \in \mathbb{R}, a > 0\}. \end{aligned}$$

So $D \subseteq X$, and the operation $*$ is the same on both sets. We check the three subgroup properties for D.

SG1 Let $(a, b), (c, d) \in D$. Then $a > 0$ and $c > 0$. We have

$$(a, b) * (c, d) = (ac, ad + b).$$

This point is in D because its first coordinate is positive (because $a > 0$ and $c > 0$). Thus D is closed under $*$.

SG2 The identity element in $(X, *)$ is $(1, 0)$. This point has a positive first coordinate, so it is in D.

SG3 Let $(a, b) \in D$. Then $a > 0$, and

$$(a, b)^{-1} = (1/a, -b/a).$$

This point has a positive first coordinate (since $a > 0$), so it is in D. Thus D contains the inverse of each of its elements.

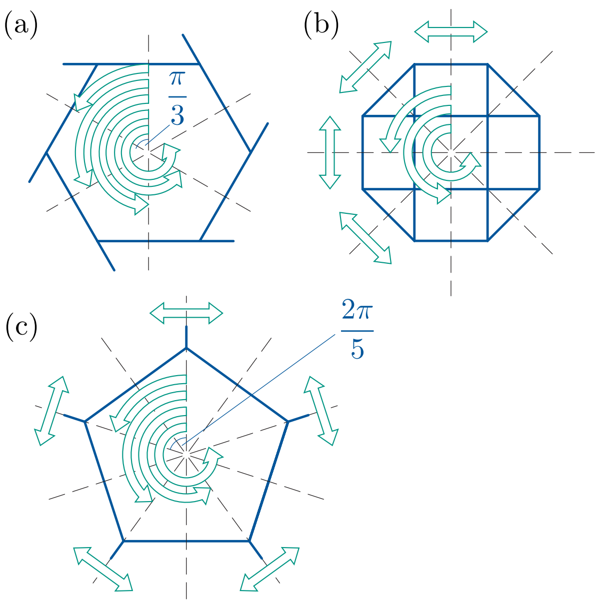
Hence $(D, *)$ satisfies the three subgroup properties, and so is a subgroup of $(X, *)$.

(c) The identity in $(X, *)$ is $(1, 0)$, which does not belong to E.

Hence property SG2 fails, so $(E, *)$ is not a subgroup of $(X, *)$.

Solution to Additional Exercise B23

The non-identity symmetries are illustrated below.



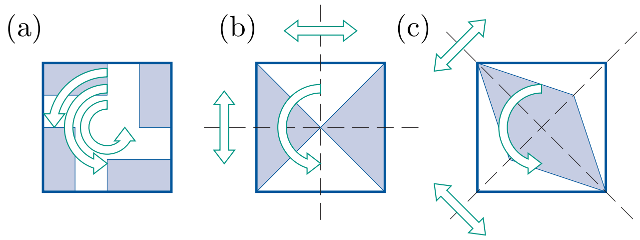
(a) The symmetries are the identity and the rotations through $\pi/3$, $2\pi/3$, π, $4\pi/3$ and $5\pi/3$.

(b) The symmetries are the identity, the rotations through $\pi/2$, π and $3\pi/2$, and the reflections in the horizontal axis, the vertical axis and the two diagonal axes of symmetry.

(c) The figure has the same symmetries as the pentagon: the identity, the rotations through multiples of $2\pi/5$ and five reflections.

Solution to Additional Exercise B24

The non-identity symmetries are illustrated below.



(a) The symmetries are the identity, and the rotations through $\pi/2$, π and $3\pi/2$.

So the symmetry group of the figure is $S^+(\square) = \{e, a, b, c\}$.

(b) The symmetries are the identity, the rotation through π, and the reflections in the horizontal and vertical axes of symmetry.

So the symmetry group of the figure is the subgroup $\{e, b, r, t\}$ of $S(\square)$.

(c) The symmetries are the identity, the rotation through π and the reflections in the two diagonal axes of symmetry.

So the symmetry group of the figure is the subgroup $\{e, b, s, u\}$ of $S(\square)$.

Solution to Additional Exercise B25

The required subgroup is

$$\left\{e, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}\right\}.$$

Solution to Additional Exercise B26

We have

$$U_{16} = \{1, 3, 5, 7, 9, 11, 13, 15\}.$$

The order of the identity element 1 is 1.

The consecutive powers of 3 are

$$\ldots, 1, 3, 9, 11, 1, 3, 9, 11, \ldots.$$

So 3 has order 4.

The element immediately before the identity element 1 in the cycle of powers of 3 is 11, so 11 is the inverse of 3 and hence it also has order 4. Also, the cycle shows that the consecutive powers of $9 = 3^2$ are

$$\ldots, 1, 9, 1, 9, 1, 9, \ldots,$$

so 9 has order 2.

The consecutive powers of 5 are

$$\ldots, 1, 5, 9, 13, 1, 5, 9, 13, \ldots.$$

So 5 has order 4.

The element immediately before the identity element 1 in the cycle of powers of 5 is 13, so 13 is the inverse of 5 and hence it also has order 4.

The consecutive powers of 7 are

$$\ldots, 1, 7, 1, 7, 1, 7, \ldots.$$

So 7 has order 2.

The consecutive powers of 15 are

$$\ldots, 1, 15, 1, 15, 1, 15, \ldots.$$

So 15 also has order 2.

In summary, the orders of the elements are as follows.

Element	1	3	5	7	9	11	13	15
Order	1	4	4	2	2	4	4	2

Solution to Additional Exercise B27

The cyclic subgroups of $(\mathbb{Z}_{14}, +_{14})$ are

$$\begin{aligned}
\langle 0\rangle &= \{0\},\\
\langle 1\rangle &= \mathbb{Z}_{14} = \langle -1\rangle = \langle 13\rangle,\\
\langle 2\rangle &= \{0,2,4,6,8,10,12\} = \langle -2\rangle = \langle 12\rangle,\\
\langle 3\rangle &= \{0,3,6,9,12,1,4,7,10,13,2,5,8,11\}\\
&= \mathbb{Z}_{14} = \langle -3\rangle = \langle 11\rangle,\\
\langle 4\rangle &= \{0,4,8,12,2,6,10\} = \langle -4\rangle = \langle 10\rangle,\\
\langle 5\rangle &= \{0,5,10,1,6,11,2,7,12,3,8,13,4,9\}\\
&= \mathbb{Z}_{14} = \langle -5\rangle = \langle 9\rangle,\\
\langle 6\rangle &= \{0,6,12,4,10,2,8\} = \langle -6\rangle = \langle 8\rangle,\\
\langle 7\rangle &= \{0,7\}.
\end{aligned}$$

The orders of the elements in $\mathbb{Z}_{14}$ are given below.

Element	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Order	1	14	7	14	7	14	7	2	7	14	7	14	7	14

Solution to Additional Exercise B28

The identity element e has order 1. It generates the trivial subgroup of $S(\pentagon)$.

Each rotation has order 5. It generates $S^+(\pentagon)$, the subgroup that contains all the rotational symmetries of the regular pentagon.

Each reflection has order 2 and generates a subgroup of order 2 containing the identity and itself.

Solution to Additional Exercise B29

(a) We have

$$U_{15} = \{1, 2, 4, 7, 8, 11, 13, 14\}.$$

The cyclic subgroups of $(U_{15}, \times_{15})$ are

$$\begin{aligned}
\langle 1\rangle &= \{1\},\\
\langle 2\rangle &= \{1,2,4,8\} = \langle 2^{-1}\rangle = \langle 8\rangle,\\
\langle 4\rangle &= \{1,4\},\\
\langle 7\rangle &= \{1,7,4,13\} = \langle 7^{-1}\rangle = \langle 13\rangle,\\
\langle 11\rangle &= \{1,11\},\\
\langle 14\rangle &= \{1,14\}.
\end{aligned}$$

No element generates the whole group, so this group is not cyclic.

(b) The cyclic subgroups of $(\mathbb{Z}_7^*, \times_7)$ are

$$\begin{aligned}
\langle 1\rangle &= \{1\},\\
\langle 2\rangle &= \{1,2,4\} = \langle 2^{-1}\rangle = \langle 4\rangle,\\
\langle 3\rangle &= \{1,3,2,6,4,5\} = \mathbb{Z}_7^* = \langle 3^{-1}\rangle = \langle 5\rangle,\\
\langle 6\rangle &= \{1,6\}.
\end{aligned}$$

Since there is an element that generates the whole group, this group is cyclic. Its generators are 3 and 5.

(c) The cyclic subgroups of $(\mathbb{Z}_{11}^*, \times_{11})$ are

$$\begin{aligned}
\langle 1\rangle &= \{1\},\\
\langle 2\rangle &= \{1,2,4,8,5,10,9,7,3,6\}\\
&= \mathbb{Z}_{11}^* = \langle 2^{-1}\rangle = \langle 6\rangle,\\
\langle 3\rangle &= \{1,3,9,5,4\} = \langle 3^{-1}\rangle = \langle 4\rangle,\\
\langle 5\rangle &= \{1,5,3,4,9\} = \langle 5^{-1}\rangle = \langle 9\rangle,\\
\langle 7\rangle &= \{1,7,5,2,3,10,4,6,9,8\}\\
&= \mathbb{Z}_{11}^* = \langle 7^{-1}\rangle = \langle 8\rangle,\\
\langle 10\rangle &= \{1,10\}.
\end{aligned}$$

Since there is an element that generates the whole group, this group is cyclic. Its generators are 2, 6, 7 and 8.

(d) We have

$$U_{14} = \{1, 3, 5, 9, 11, 13\}.$$

The cyclic subgroups of $(U_{14}, \times_{14})$ are

$$\begin{aligned}
\langle 1\rangle &= \{1\},\\
\langle 3\rangle &= \{1, 3, 9, 13, 11, 5\} = \langle 3^{-1}\rangle = \langle 5\rangle,\\
\langle 9\rangle &= \{1, 9, 11\} = \langle 9^{-1}\rangle = \langle 11\rangle,\\
\langle 13\rangle &= \{1, 13\}.
\end{aligned}$$

Since there is an element that generates the whole group, this group is cyclic. Its generators are 3 and 5.

Solution to Additional Exercise B30

(a) $r_{2\pi/9}$ has order 9.

(b) $r_{3\pi/7}$ has order 14.

(c) r_3 has infinite order.

Solution to Additional Exercise B31

(a) The HCF of 3 and 15 is 3, so the order of 3 in $(\mathbb{Z}_{15}, +_{15})$ is $15/3 = 5$.

(Check: In the group $(\mathbb{Z}_{15}, +_{15})$,

$$\langle 3\rangle = \{0, 3, 6, 9, 12\},$$

so 3 has order 5.)

(b) The HCF of 9 and 21 is 3, so the order of 9 in $(\mathbb{Z}_{21}, +_{21})$ is $21/3 = 7$.

(Check: In the group $(\mathbb{Z}_{21}, +_{21})$,

$$\langle 9\rangle = \{0, 9, 18, 6, 15, 3, 12\},$$

so 9 has order 7.)

Solution to Additional Exercise B32

(a) The generators of $(\mathbb{Z}_{14}, +_{14})$ are the integers in $\mathbb{Z}_{14}$ that are coprime to 14 (by Corollary B40), namely,

1, 3, 5, 9, 11 and 13.

(b) The generators of $(\mathbb{Z}_{16}, +_{16})$ are the integers in $\mathbb{Z}_{16}$ that are coprime to 16, (by Corollary B40), namely,

1, 3, 5, 7, 9, 11, 13 and 15.

Solution to Additional Exercise B33

We have

$$U_{24} = \{1, 5, 7, 11, 13, 17, 19, 23\}.$$

We find the cyclic subgroup generated by each element:

$$\begin{aligned}
\langle 1\rangle &= \{1\},\\
\langle 5\rangle &= \{1, 5\},\\
\langle 7\rangle &= \{1, 7\},\\
\langle 11\rangle &= \{1, 11\},\\
\langle 13\rangle &= \{1, 13\},\\
\langle 17\rangle &= \{1, 17\},\\
\langle 19\rangle &= \{1, 19\},\\
\langle 23\rangle &= \{1, 23\}.
\end{aligned}$$

(This group has a large number of non-cyclic subgroups of order 4: for example, $\{1, 5, 7, 11\}$ and $\{1, 5, 13, 17\}$.)

Solution to Additional Exercise B34

In $(\mathbb{Z}_{13}^*, \times_{13})$,

$$\begin{aligned}
\langle 1\rangle &= \{1\},\\
\langle 2\rangle &= \{1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7\}\\
&= \mathbb{Z}_{13}^* = \langle 2^{-1}\rangle = \langle 7\rangle,\\
\langle 3\rangle &= \{1, 3, 9\} = \langle 3^{-1}\rangle = \langle 9\rangle,\\
\langle 4\rangle &= \{1, 4, 3, 12, 9, 10\} = \langle 4^{-1}\rangle = \langle 10\rangle,\\
\langle 5\rangle &= \{1, 5, 12, 8\} = \langle 5^{-1}\rangle = \langle 8\rangle,\\
\langle 6\rangle &= \{1, 6, 10, 8, 9, 2, 12, 7, 3, 5, 4, 11\}\\
&= \mathbb{Z}_{13}^* = \langle 6^{-1}\rangle = \langle 11\rangle,\\
\langle 12\rangle &= \{1, 12\}.
\end{aligned}$$

Since there is an element that generates the whole group, this group is cyclic. Its generators are 2, 6, 7 and 11. By Theorem B36, all the subgroups of $(\mathbb{Z}_{13}^*, \times_{13})$ are cyclic, so the list above gives all the subgroups of this group.

Solution to Additional Exercise B35

(a) The Cayley tables are as follows.

$\times_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

$\times_8$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

The Cayley tables have the same pattern, so, by matching the elements in the order in which they appear in the borders, we obtain the following isomorphism:

$$\begin{aligned} \phi : U_{12} &\longrightarrow U_8 \\ 1 &\longmapsto 1 \\ 5 &\longmapsto 3 \\ 7 &\longmapsto 5 \\ 11 &\longmapsto 7. \end{aligned}$$

(There are several such isomorphisms: see Additional Exercise B37.)

(b) By the solution to Additional Exercise B29(c), $(\mathbb{Z}_{11}^*, \times_{11})$ is cyclic with generators 2, 6, 7 and 8. Also, $(\mathbb{Z}_{10}, +_{10})$ is cyclic and generated by 1. Using Strategy B6 to match powers of 2, 6, 7 and 8 in $(\mathbb{Z}_{11}^*, \times_{11})$ with multiples of 1 in $(\mathbb{Z}_{10}, +_{10})$ gives the following four isomorphisms, respectively. (You were asked for only one of these.)

$$\begin{array}{lll} \phi_1 : \mathbb{Z}_{11}^* & \longrightarrow & \mathbb{Z}_{10} \\ 2^0 = \ \ 1 & \longmapsto & 0 \\ 2^1 = \ \ 2 & \longmapsto & 1 \\ 2^2 = \ \ 4 & \longmapsto & 2 \\ 2^3 = \ \ 8 & \longmapsto & 3 \\ 2^4 = \ \ 5 & \longmapsto & 4 \\ 2^5 = 10 & \longmapsto & 5 \\ 2^6 = \ \ 9 & \longmapsto & 6 \\ 2^7 = \ \ 7 & \longmapsto & 7 \\ 2^8 = \ \ 3 & \longmapsto & 8 \\ 2^9 = \ \ 6 & \longmapsto & 9 \end{array} \qquad \begin{array}{lll} \phi_2 : \mathbb{Z}_{11}^* & \longrightarrow & \mathbb{Z}_{10} \\ 6^0 = \ \ 1 & \longmapsto & 0 \\ 6^1 = \ \ 6 & \longmapsto & 1 \\ 6^2 = \ \ 3 & \longmapsto & 2 \\ 6^3 = \ \ 7 & \longmapsto & 3 \\ 6^4 = \ \ 9 & \longmapsto & 4 \\ 6^5 = 10 & \longmapsto & 5 \\ 6^6 = \ \ 5 & \longmapsto & 6 \\ 6^7 = \ \ 8 & \longmapsto & 7 \\ 6^8 = \ \ 4 & \longmapsto & 8 \\ 6^9 = \ \ 2 & \longmapsto & 9 \end{array}$$

$$\begin{array}{lll} \phi_3 : \mathbb{Z}_{11}^* & \longrightarrow & \mathbb{Z}_{10} \\ 7^0 = \ \ 1 & \longmapsto & 0 \\ 7^1 = \ \ 7 & \longmapsto & 1 \\ 7^2 = \ \ 5 & \longmapsto & 2 \\ 7^3 = \ \ 2 & \longmapsto & 3 \\ 7^4 = \ \ 3 & \longmapsto & 4 \\ 7^5 = 10 & \longmapsto & 5 \\ 7^6 = \ \ 4 & \longmapsto & 6 \\ 7^7 = \ \ 6 & \longmapsto & 7 \\ 7^8 = \ \ 9 & \longmapsto & 8 \\ 7^9 = \ \ 8 & \longmapsto & 9 \end{array} \qquad \begin{array}{lll} \phi_4 : \mathbb{Z}_{11}^* & \longrightarrow & \mathbb{Z}_{10} \\ 8^0 = \ \ 1 & \longmapsto & 0 \\ 8^1 = \ \ 8 & \longmapsto & 1 \\ 8^2 = \ \ 9 & \longmapsto & 2 \\ 8^3 = \ \ 6 & \longmapsto & 3 \\ 8^4 = \ \ 4 & \longmapsto & 4 \\ 8^5 = 10 & \longmapsto & 5 \\ 8^6 = \ \ 3 & \longmapsto & 6 \\ 8^7 = \ \ 2 & \longmapsto & 7 \\ 8^8 = \ \ 5 & \longmapsto & 8 \\ 8^9 = \ \ 7 & \longmapsto & 9 \end{array}$$

(c) We have

$$U_9 = \{1, 2, 4, 5, 7, 8\},$$

$$U_{14} = \{1, 3, 5, 9, 11, 13\}.$$

By the solution to Exercise B79, $(U_9, \times_9)$ is cyclic with generators 2 and 5. Also, by the solution to Additional Exercise B29(e), $(U_{14}, \times_{14})$ is cyclic with generators 3 and 5.

Using Strategy B6 to match powers of the generators 2 and 5 in $(U_9, \times_9)$ with powers of the generator 3 in $(U_{14}, \times_{14})$ gives the following two isomorphisms, respectively. (You were asked for only one of these.)

$$\begin{array}{rcl} \phi_1 : U_9 & \longrightarrow & U_{14} \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 3 \\ 2^2 & \longmapsto & 3^2 \\ 2^3 & \longmapsto & 3^3 \\ 2^4 & \longmapsto & 3^4 \\ 2^5 & \longmapsto & 3^5, \end{array} \qquad \begin{array}{rcl} \phi_2 : U_9 & \longrightarrow & U_{14} \\ 1 & \longmapsto & 1 \\ 5 & \longmapsto & 3 \\ 5^2 & \longmapsto & 3^2 \\ 5^3 & \longmapsto & 3^3 \\ 5^4 & \longmapsto & 3^4 \\ 5^5 & \longmapsto & 3^5. \end{array}$$

These isomorphisms simplify to the following.

$$\begin{array}{rcl} \phi_1 : U_9 & \longrightarrow & U_{14} \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 3 \\ 4 & \longmapsto & 9 \\ 8 & \longmapsto & 13 \\ 7 & \longmapsto & 11 \\ 5 & \longmapsto & 5, \end{array} \qquad \begin{array}{rcl} \phi_2 : U_9 & \longrightarrow & U_{14} \\ 1 & \longmapsto & 1 \\ 5 & \longmapsto & 3 \\ 7 & \longmapsto & 9 \\ 8 & \longmapsto & 13 \\ 4 & \longmapsto & 11 \\ 2 & \longmapsto & 5. \end{array}$$

Solution to Additional Exercise B36

Both groups have order 8.

However, the solution to Additional Exercise B26 shows that $(U_{16}, \times_{16})$ has exactly three elements of order 2, whereas the solution to Additional Exercise B33 shows that $(U_{24}, \times_{24})$ has seven elements of order 2. It follows that these two groups are not isomorphic.

Solution to Additional Exercise B37

Since $(U_{12}, \times_{12})$ and $(U_8, \times_8)$ *are* isomorphic, the number of isomorphisms from $(U_{12}, \times_{12})$ to $(U_8, \times_8)$ is equal to the number of ways in which we can rearrange the Cayley table of $(U_8, \times_8)$ without changing its pattern (including the 'trivial rearrangement', for which the table remains the same).

Changing the position of the identity element in the table borders *will* change the pattern, since there is only one element with the defining property of the identity element.

However, no matter how we rearrange the positions of the other three elements (keeping them the same in both borders, of course), the pattern of the table will not change. This is because in $(U_8, \times_8)$ (and in any Klein four-group), composing each of the three non-identity elements with itself gives the identity element, and composing any of the three non-identity elements with a different non-identity element gives the third non-identity element.

There are 3 ways to choose the non-identity element to appear first in the borders, 2 ways to choose the second, and then just 1 way to choose the third, so there are $3 \times 2 \times 1 = 6$ arrangements of the three non-identity elements. Hence there are 6 isomorphisms from $(U_{12}, \times_{12})$ to $(U_8, \times_8)$.

Solution to Additional Exercise B38

Consider any set of groups. We check that the relation of 'is isomorphic to' is reflexive (property E1), symmetric (property E2) and transitive (property E3) on this set.

E1 Let $(G, \circ)$ be any group in the set. Let ϕ be the mapping given by

$$\begin{aligned} \phi : G &\longrightarrow G \\ x &\longmapsto x \quad (x \in G). \end{aligned}$$

Then ϕ is one-to-one and onto, and for all $x \in G$,

$$\phi(x \circ y) = x \circ y = \phi(x) \circ \phi(y),$$

so ϕ is an isomorphism. Hence $(G, \circ) \cong (G, \circ)$. Thus 'is isomorphic to' is reflexive.

E2 Let $(G, \circ)$ and $(H, *)$ be any groups in the set, and suppose that $(G, \circ) \cong (H, *)$. Then there is an isomorphism $\phi : G \longrightarrow H$. Since ϕ is one-to-one and onto, it has an inverse mapping $\phi^{-1} : H \longrightarrow G$, which is also one-to-one and onto. Let h_1 and h_2 be any elements of H. Then $h_1 = \phi(g_1)$ and $h_2 = \phi(g_2)$, where $g_1, g_2 \in G$. So

$$\begin{aligned} &\phi^{-1}(h_1 * h_2) \\ &= \phi^{-1}(\phi(g_1) * \phi(g_2)) \\ &= \phi^{-1}(\phi(g_1 \circ g_2)) \\ &\qquad\qquad \text{(since } \phi \text{ is an isomorphism)} \\ &= g_1 \circ g_2 \\ &= \phi^{-1}(h_1) \circ \phi^{-1}(h_2). \end{aligned}$$

Hence ϕ^{-1} is an isomorphism, so $(H, *) \cong (G, \circ)$. Thus 'is isomorphic to' is symmetric.

E3 Let $(G, \bullet)$, $(H, *)$ and $(K, \vartriangle)$ be any groups in the set, and suppose that $(G, \bullet) \cong (H, *)$ and $(H, *) \cong (K, \vartriangle)$. Then there exist isomorphisms $\phi : G \longrightarrow H$ and $\psi : H \longrightarrow K$. Since both ϕ and ψ are one-to-one and onto, so is their composite $\psi \circ \phi$. Let x and y be any elements of G. Then

$$\begin{aligned} &(\psi \circ \phi)(x \bullet y) \\ &= \psi(\phi(x \bullet y)) \\ &= \psi(\phi(x) * \phi(y))) \\ &\qquad\qquad \text{(since } \phi \text{ is an isomorphism)} \\ &= \psi(\phi(x)) \vartriangle \psi(\phi(y)) \\ &\qquad\qquad \text{(since } \psi \text{ is an isomorphism)} \\ &= (\psi \circ \phi)(x) \vartriangle (\psi \circ \phi)(y). \end{aligned}$$

Hence $\psi \circ \phi$ is an isomorphism, so $(G, \bullet) \cong (K, \vartriangle)$. Thus 'is isomorphic to' is transitive.

Since the relation 'is isomorphic to' is reflexive, symmetric and transitive, it is an equivalence relation.

Additional exercises for Unit B3

Section 1

Additional Exercise B39

Express the following permutations, which are given in two-line form, in cycle form.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 2 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 1 & 3 & 7 & 2 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 7 & 6 & 5 & 8 & 3 & 1 & 2 \end{pmatrix}$

Additional Exercise B40

Let f, g and h be the following permutations in S_5:

$f = (1\ 4\ 5)$,

$g = (1\ 2)(3\ 4)$,

$h = (1\ 3\ 4\ 2\ 5)$.

Write down each of the following as a permutation in cycle form:

(a) $g \circ f$ (b) $g \circ h$ (c) $h \circ g \circ f$

(d) h^{-1} (e) $(g \circ h)^{-1}$ (f) h^2 (g) h^3

Additional Exercise B41

Determine the cycle form of each of the following permutations in S_8.

(a) $(1\ 7\ 6\ 2)(3\ 5\ 4\ 8) \circ (2\ 8\ 7\ 3\ 5) \circ (1\ 6\ 2)(5\ 3)$

(b) $(1\ 3)(2\ 5) \circ (3\ 6)(4\ 5) \circ (3\ 4)(7\ 8) \circ (1\ 6)(2\ 3)$

Section 2

Additional Exercise B42

Determine the order of each of the following elements of S_7.

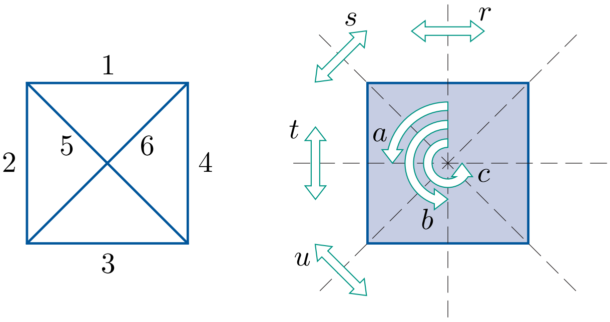
(a) $f = (1\ 2\ 4\ 6)(3\ 5)$

(b) $g = (1\ 3)(2\ 7)(4\ 6)$

(c) $h = (1\ 2\ 3)(4\ 7)$

Additional Exercise B43

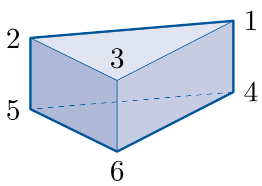
By labelling the four edges and two diagonals of the square as shown on the left below, find a subgroup of S_6 that is isomorphic to $S(\square)$.



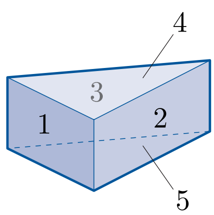
Additional Exercise B44

The solid in parts (a) and (b) below is a prism with two faces that are isosceles triangles and three faces that are rectangles. The edge joining the vertices labelled 2 and 3 in part (a) is the shortest edge of the top face.

(a) Write down the permutations in S_6 that represent the symmetries of the solid when its vertices are labelled as shown below.



(b) Write down the permutations in S_5 that represent the symmetries of the solid when its faces are labelled as shown below. (In the diagram faces 1, 2 and 3 are vertical and faces 4 and 5 are horizontal.)



Section 3

Additional Exercise B45

Express each of the following permutations in S_9 as a composite of transpositions.

(a) $(1\ 7\ 9\ 4)(3\ 5\ 8\ 6)$ (b) $(1\ 8\ 2)(3\ 7)(4\ 6\ 9)$

Additional Exercise B46

Determine the parity of each of the following permutations.

(a) $(1\ 2\ 3)(4\ 5\ 6\ 7)$ (b) $(1\ 2\ 3)(4\ 5)(6\ 7\ 8\ 9)$

(c) $(1\ 2\ 3\ 5) \circ (1\ 5\ 2)$

(d) $(2\ 3\ 4)(5\ 6) \circ (1\ 3\ 5\ 6\ 2)$

Additional Exercise B47

List the possible cycle structures of S_6. Write down a representative element for each cycle structure and indicate whether the element is an even or an odd permutation.

Additional Exercise B48

Give examples of the following.

(a) Two even permutations in S_6, each having order 3, but with different cycle structures.

(b) An even and an odd permutation in S_6, each with order 4.

(c) Two odd permutations and an even permutation in S_7, each having order 6, but with different cycle structures.

Section 4

Additional Exercise B49

(a) Find all the permutations g in S_5 such that

$$g \circ (1\ 2\ 3\ 4\ 5) \circ g^{-1} = (1\ 3\ 4\ 2\ 5).$$

(b) Find all the permutations g in S_5 such that

$$g \circ (1\ 2\ 3\ 5) \circ g^{-1} = (1\ 2\ 3\ 5).$$

(c) Find all the permutations g in S_6 such that

$$g \circ (1\ 2\ 3)(4\ 5) \circ g^{-1} = (2\ 3\ 4)(5\ 6).$$

(d) Find all the permutations g in S_4 such that

$$g \circ (1\ 2) \circ g^{-1} = (3\ 4).$$

Additional Exercise B50

The following set is a subgroup of S_5:

$$H = \{e, (1\ 2)(3\ 4\ 5), (3\ 5\ 4), (1\ 2), (3\ 4\ 5), (1\ 2)(3\ 5\ 4)\}.$$

(It is the cyclic subgroup generated by the permutation $(1\ 2)(3\ 4\ 5)$.)

Determine the following conjugate subgroups.

(a) $(2\ 3) \circ H \circ (2\ 3)^{-1}$

(b) $(1\ 2\ 5) \circ H \circ (1\ 2\ 5)^{-1}$

Additional Exercise B51

Each of the following sets is a subgroup of S_5:

$$H = \{e, (1\ 4), (1\ 5), (4\ 5), (1\ 4\ 5), (1\ 5\ 4)\},$$
$$K = \{e, (2\ 3), (2\ 5), (3\ 5), (2\ 3\ 5), (2\ 5\ 3)\}.$$

(You are not asked to show this.)

Find a permutation f in S_5 such that $K = f \circ H \circ f^{-1}$, and a permutation g in S_5 such that $H = g \circ K \circ g^{-1}$.

Section 5

Additional Exercise B52

(a) Write down an element of S_5 of order 6 and hence find a cyclic subgroup of S_5 of order 6.

(b) Given that S_5 contains exactly 20 permutations of order 6, find the number of cyclic subgroups of S_5 of order 6.

Additional Exercise B53

By considering a regular pentagon, find two different non-cyclic subgroups of S_5 of order 10.

Section 6

Additional Exercise B54

Find a permutation group that is isomorphic to the group with the following group table.

$\circ$	e	a	p	x	q	y	r	z
e	e	a	p	x	q	y	r	z
a	a	e	x	p	y	q	z	r
p	p	x	a	e	r	z	y	q
x	x	p	e	a	z	r	q	y
q	q	y	z	r	a	e	p	x
y	y	q	r	z	e	a	x	p
r	r	z	q	y	x	p	a	e
z	z	r	y	q	p	x	e	a

Solutions to additional exercises for Unit B3

Solution to Additional Exercise B39

We use Strategy B7: we trace the images of the symbols in the order in which they are encountered, starting at the symbol 1.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 2 & 1 \end{pmatrix} = (1\ 4\ 5\ 2\ 6)(3)$
$= (1\ 4\ 5\ 2\ 6).$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 1 & 3 & 7 & 2 & 5 \end{pmatrix} = (1\ 4\ 3)(2\ 6)(5\ 7).$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 7 & 6 & 5 & 8 & 3 & 1 & 2 \end{pmatrix} = (1\ 4\ 5\ 8\ 2\ 7)(3\ 6).$

Solution to Additional Exercise B40

For parts (a)–(c), we use Strategy B8. For parts (d) and (e), we use Strategy B9 (that is, we calculate inverses by writing each cycle in reverse order).

(a) $g \circ f = (1\ 2)(3\ 4) \circ (1\ 4\ 5) = (1\ 3\ 4\ 5\ 2)$

(b) $g \circ h = (1\ 2)(3\ 4) \circ (1\ 3\ 4\ 2\ 5) = (1\ 4)(2\ 5)$

(c) To find $h \circ g \circ f$, we find $h \circ (g \circ f)$ using the permutation $g \circ f$ that we have already found:

$$\begin{aligned} h \circ g \circ f &= h \circ (g \circ f) \\ &= (1\ 3\ 4\ 2\ 5) \circ (1\ 3\ 4\ 5\ 2) \\ &= (1\ 4)(2\ 3)(5) \\ &= (1\ 4)(2\ 3). \end{aligned}$$

(d) Since

$$h = (1\ 3\ 4\ 2\ 5),$$

we have

$$h^{-1} = (5\ 2\ 4\ 3\ 1) = (1\ 5\ 2\ 4\ 3).$$

(e) Since

$$g \circ h = (1\ 4)(2\ 5),$$

we have

$$(g \circ h)^{-1} = (4\ 1)(5\ 2) = g \circ h.$$

(Note that $g \circ h$ is a composite of transpositions, so $g \circ h$ is self-inverse.)

(f) To find h^2, we map each symbol in h to the symbol two places around the cycle:

$$h^2 = (1\ 4\ 5\ 3\ 2).$$

(g) To find h^3, we map each symbol in h to the symbol three places around the cycle:

$$h^3 = (1\ 2\ 3\ 5\ 4).$$

Solution to Additional Exercise B41

We use Strategy B8.

(a) We have

$$\begin{aligned} &(1\ 7\ 6\ 2)(3\ 5\ 4\ 8) \circ (2\ 8\ 7\ 3\ 5) \circ (1\ 6\ 2)(5\ 3) \\ &= (1\ 2\ 7\ 5\ 4\ 8\ 6\ 3). \end{aligned}$$

(b) We have

$$\begin{aligned} &(1\ 3)(2\ 5) \circ (3\ 6)(4\ 5) \circ (3\ 4)(7\ 8) \circ (1\ 6)(2\ 3) \\ &= (1)(2)(3\ 5\ 4\ 6)(7\ 8) \\ &= (3\ 5\ 4\ 6)(7\ 8). \end{aligned}$$

Solution to Additional Exercise B42

By Theorem B55, the order of a permutation is the least common multiple of the lengths of its cycles.

(a) The cycles of f have lengths 4 and 2, so f has order 4.

(b) The cycles of g have lengths 2, 2 and 2, so g has order 2.

(c) The cycles of h have lengths 3 and 2, so h has order 6.

Solution to Additional Exercise B43

The symmetries of $S(\square)$ applied to the labelled square correspond to the following permutations of $\{1, 2, 3, 4, 5, 6\}$.

Symmetry	Permutation
e	e
a	$(1\ 2\ 3\ 4)(5\ 6)$
b	$(1\ 3)(2\ 4)$
c	$(1\ 4\ 3\ 2)(5\ 6)$
r	$(2\ 4)(5\ 6)$
s	$(1\ 2)(3\ 4)$
t	$(1\ 3)(5\ 6)$
u	$(1\ 4)(2\ 3)$

Hence a subgroup of S_6 that is isomorphic to $S(\square)$ is

$$\begin{aligned} &\{e, (1\ 2\ 3\ 4)(5\ 6), (1\ 3)(2\ 4), (1\ 4\ 3\ 2)(5\ 6), \\ &\ (2\ 4)(5\ 6), (1\ 2)(3\ 4), (1\ 3)(5\ 6), (1\ 4)(2\ 3)\}. \end{aligned}$$

Solution to Additional Exercise B44

(a) The symmetries of the labelled solid are represented by the following permutations in S_6:

$$e,\ (2\ 3)(5\ 6),\ (1\ 4)(2\ 5)(3\ 6),\ (1\ 4)(2\ 6)(3\ 5).$$

(b) The symmetries of the labelled solid are represented by the following permutations in S_5:

$$e,\ (2\ 3),\ (4\ 5),\ (2\ 3)(4\ 5).$$

Solution to Additional Exercise B45

(a) $(1\ 7\ 9\ 4)(3\ 5\ 8\ 6)$
$$= (1\ 7\ 9\ 4) \circ (3\ 5\ 8\ 6)$$
$$= (1\ 4) \circ (1\ 9) \circ (1\ 7) \circ (3\ 6) \circ (3\ 8) \circ (3\ 5)$$

(b) $(1\ 8\ 2)(3\ 7)(4\ 6\ 9)$
$$= (1\ 8\ 2) \circ (3\ 7) \circ (4\ 6\ 9)$$
$$= (1\ 2) \circ (1\ 8) \circ (3\ 7) \circ (4\ 9) \circ (4\ 6)$$

Solution to Additional Exercise B46

We determine the parity of each permutation using Strategy B11.

(a) $(1\ 2\ 3)(4\ 5\ 6\ 7)$ is

$$\text{even} + \text{odd} = \text{odd},$$

so this is an odd permutation.

(b) $(1\ 2\ 3)(4\ 5)(6\ 7\ 8\ 9)$ is

$$\text{even} + \text{odd} + \text{odd} = \text{even},$$

so this is an even permutation.

(c) $(1\ 2\ 3\ 5) \circ (1\ 5\ 2)$ is

$$\text{odd} + \text{even} = \text{odd},$$

so this is an odd permutation.

(d) $(2\ 3\ 4)(5\ 6) \circ (1\ 3\ 5\ 6\ 2)$ is

$$\text{even} + \text{odd} + \text{even} = \text{odd},$$

so this is an odd permutation.

Solution to Additional Exercise B47

The cycle structures, representative elements and parities are listed in the following table.

Cycle structure	Element of S_6	Parity
e	e	even
$(-\ -)$	$(1\ 2)$	odd
$(-\ -\ -)$	$(1\ 2\ 3)$	even
$(-\ -\ -\ -)$	$(1\ 2\ 3\ 4)$	odd
$(-\ -\ -\ -\ -)$	$(1\ 2\ 3\ 4\ 5)$	even
$(-\ -\ -\ -\ -\ -)$	$(1\ 2\ 3\ 4\ 5\ 6)$	odd
$(-\ -)\ (-\ -)$	$(1\ 2)(3\ 4)$	even
$(-\ -)\ (-\ -)\ (-\ -)$	$(1\ 2)(3\ 4)(5\ 6)$	odd
$(-\ -)(-\ -\ -)$	$(1\ 2)(3\ 4\ 5)$	odd
$(-\ -)\ (-\ -\ -\ -)$	$(1\ 2)(3\ 4\ 5\ 6)$	even
$(-\ -\ -)\ (-\ -\ -)$	$(1\ 2\ 3)(4\ 5\ 6)$	even

Solution to Additional Exercise B48

(a) $(1\ 2\ 3)$ and $(1\ 2\ 3)(4\ 5\ 6)$ are both even permutations in S_6, with order 3. They have different cycle structures.

(b) $(1\ 2\ 3\ 4)$ is an odd permutation and $(1\ 2\ 3\ 4)(5\ 6)$ is an even permutation in S_6; each has order 4.

(c) $(1\ 2\ 3\ 4\ 5\ 6)$ and $(1\ 2)(3\ 4\ 5)$ are both odd permutations in S_7, and $(1\ 2)(3\ 4)(5\ 6\ 7)$ is an even permutation in S_7; all three have order 6.

Solution to Additional Exercise B49

We use Strategy B12.

(a) Here $x = (1\ 2\ 3\ 4\ 5)$ and $y = (1\ 3\ 4\ 2\ 5)$.

Writing y as $(1\ 3\ 4\ 2\ 5)$ and matching up the cycles, we obtain

$$\begin{array}{rl} x = & (1\ 2\ 3\ 4\ 5) \\ g\downarrow & \ \ \downarrow\downarrow\downarrow\downarrow\downarrow \\ y = & (1\ 3\ 4\ 2\ 5) \end{array}, \text{ which gives } g = (2\ 3\ 4).$$

Similarly, for the other four ways of writing y, we obtain

$$\begin{array}{rl} x = & (1\ 2\ 3\ 4\ 5) \\ g\downarrow & \ \ \downarrow\downarrow\downarrow\downarrow\downarrow \\ y = & (3\ 4\ 2\ 5\ 1) \end{array}, \text{ which gives } g = (1\ 3\ 2\ 4\ 5);$$

$$\begin{array}{rl} x = & (1\ 2\ 3\ 4\ 5) \\ g\downarrow & \ \ \downarrow\downarrow\downarrow\downarrow\downarrow \\ y = & (4\ 2\ 5\ 1\ 3) \end{array}, \text{ which gives } g = (1\ 4)(3\ 5);$$

$$\begin{array}{rl} x = & (1\ 2\ 3\ 4\ 5) \\ g\downarrow & \ \ \downarrow\downarrow\downarrow\downarrow\downarrow \\ y = & (2\ 5\ 1\ 3\ 4) \end{array}, \text{ which gives } g = (1\ 2\ 5\ 4\ 3);$$

$$\begin{array}{rl} x = & (1\ 2\ 3\ 4\ 5) \\ g\downarrow & \ \ \downarrow\downarrow\downarrow\downarrow\downarrow \\ y = & (5\ 1\ 3\ 4\ 2) \end{array}, \text{ which gives } g = (1\ 5\ 2).$$

(b) Here $x = y = (1\ 2\ 3\ 5)$.

Matching up the cycles for each of the four ways of writing $(1\ 2\ 3\ 5)$, we obtain

$$\begin{aligned} x &= (1\ 2\ 3\ 5)\\ g\downarrow\ & \quad\ \downarrow\downarrow\downarrow\downarrow\ \text{, which gives } g = e;\\ y &= (1\ 2\ 3\ 5)\end{aligned}$$

$$\begin{aligned} x &= (1\ 2\ 3\ 5)\\ g\downarrow\ & \quad\ \downarrow\downarrow\downarrow\downarrow\ \text{, which gives } g = (1\ 2\ 3\ 5);\\ y &= (2\ 3\ 5\ 1)\end{aligned}$$

$$\begin{aligned} x &= (1\ 2\ 3\ 5)\\ g\downarrow\ & \quad\ \downarrow\downarrow\downarrow\downarrow\ \text{, which gives } g = (1\ 3)(2\ 5);\\ y &= (3\ 5\ 1\ 2)\end{aligned}$$

$$\begin{aligned} x &= (1\ 2\ 3\ 5)\\ g\downarrow\ & \quad\ \downarrow\downarrow\downarrow\downarrow\ \text{, which gives } g = (1\ 5\ 3\ 2).\\ y &= (5\ 1\ 2\ 3)\end{aligned}$$

(c) Here $x = (1\ 2\ 3)(4\ 5)$ and $y = (2\ 3\ 4)(5\ 6)$.

There are six ways to match up the cycles, as follows:

$$\begin{aligned} x &= (1\ 2\ 3)(4\ 5)(6)\\ g\downarrow\ & \quad\ \downarrow\downarrow\downarrow\ \ \downarrow\downarrow\ \ \downarrow\text{, giving } g = (1\ 2\ 3\ 4\ 5\ 6);\\ y &= (2\ 3\ 4)(5\ 6)(1)\end{aligned}$$

$$\begin{aligned} x &= (1\ 2\ 3)(4\ 5)(6)\\ g\downarrow\ & \quad\ \downarrow\downarrow\downarrow\ \ \downarrow\downarrow\ \ \downarrow\text{, giving } g = (1\ 3\ 2\ 4\ 6);\\ y &= (3\ 4\ 2)(6\ 5)(1)\end{aligned}$$

$$\begin{aligned} x &= (1\ 2\ 3)(4\ 5)(6)\\ g\downarrow\ & \quad\ \downarrow\downarrow\downarrow\ \ \downarrow\downarrow\ \ \downarrow\text{, giving } g = (1\ 4\ 5\ 6);\\ y &= (4\ 2\ 3)(5\ 6)(1)\end{aligned}$$

$$\begin{aligned} x &= (1\ 2\ 3)(4\ 5)(6)\\ g\downarrow\ & \quad\ \downarrow\downarrow\downarrow\ \ \downarrow\downarrow\ \ \downarrow\text{, giving } g = (1\ 2\ 3\ 4\ 6);\\ y &= (2\ 3\ 4)(6\ 5)(1)\end{aligned}$$

$$\begin{aligned} x &= (1\ 2\ 3)(4\ 5)(6)\\ g\downarrow\ & \quad\ \downarrow\downarrow\downarrow\ \ \downarrow\downarrow\ \ \downarrow\text{, giving } g = (1\ 3\ 2\ 4\ 5\ 6);\\ y &= (3\ 4\ 2)(5\ 6)(1)\end{aligned}$$

$$\begin{aligned} x &= (1\ 2\ 3)(4\ 5)(6)\\ g\downarrow\ & \quad\ \downarrow\downarrow\downarrow\ \ \downarrow\downarrow\ \ \downarrow\text{, giving } g = (1\ 4\ 6).\\ y &= (4\ 2\ 3)(6\ 5)(1)\end{aligned}$$

(d) Here $x = (1\ 2)$ and $y = (3\ 4)$.

There are four ways to match up the cycles, as follows:

$$\begin{aligned} x &= (1\ 2)(3)(4)\\ g\downarrow\ & \quad\ \downarrow\downarrow\ \ \downarrow\ \ \downarrow\text{, giving } g = (1\ 3)(2\ 4);\\ y &= (3\ 4)(1)(2)\end{aligned}$$

$$\begin{aligned} x &= (1\ 2)(3)(4)\\ g\downarrow\ & \quad\ \downarrow\downarrow\ \ \downarrow\ \ \downarrow\text{, giving } g = (1\ 3\ 2\ 4);\\ y &= (3\ 4)(2)(1)\end{aligned}$$

$$\begin{aligned} x &= (1\ 2)(3)(4)\\ g\downarrow\ & \quad\ \downarrow\downarrow\ \ \downarrow\ \ \downarrow\text{, giving } g = (1\ 4\ 2\ 3);\\ y &= (4\ 3)(1)(2)\end{aligned}$$

$$\begin{aligned} x &= (1\ 2)(3)(4)\\ g\downarrow\ & \quad\ \downarrow\downarrow\ \ \downarrow\ \ \downarrow\text{, giving } g = (1\ 4)(2\ 3).\\ y &= (4\ 3)(2)(1)\end{aligned}$$

Solution to Additional Exercise B50

Here H is

$$\{e,\ (1\ 2)(3\ 4\ 5),\ (3\ 5\ 4),\ (1\ 2),\ (3\ 4\ 5),\ (1\ 2)(3\ 5\ 4)\}.$$

(a) We obtain the elements of $(2\ 3) \circ H \circ (2\ 3)^{-1}$ by interchanging the symbols 2 and 3:

$$\{e,\ (1\ 3)(2\ 4\ 5),\ (2\ 5\ 4),\ (1\ 3),\ (2\ 4\ 5),\ (1\ 3)(2\ 5\ 4)\}.$$

(Note that H is the cyclic subgroup $\langle(1\ 2)(3\ 4\ 5)\rangle$ and that $(2\ 3) \circ H \circ (2\ 3)^{-1}$ is the cyclic subgroup $\langle(1\ 3)(2\ 4\ 5)\rangle$; a generator of the conjugate subgroup is obtained by interchanging 2 and 3 in a generator of H.)

(b) We obtain the elements of $(1\ 2\ 5) \circ H \circ (1\ 2\ 5)^{-1}$ by keeping 3 and 4 fixed and replacing 1 by 2, 2 by 5 and 5 by 1:

$$\{e,\ (2\ 5)(3\ 4\ 1),\ (3\ 1\ 4),\ (2\ 5),\ (3\ 4\ 1),\ (2\ 5)(3\ 1\ 4)\};$$

that is,

$$\{e,\ (1\ 3\ 4)(2\ 5),\ (1\ 4\ 3),\ (2\ 5),\ (1\ 3\ 4),\ (1\ 4\ 3)(2\ 5)\}.$$

(Note that H is the cyclic subgroup $\langle(1\ 2)(3\ 4\ 5)\rangle$ and that $(1\ 2\ 5) \circ H \circ (1\ 2\ 5)^{-1}$ is the cyclic subgroup $\langle(2\ 5)(3\ 4\ 1)\rangle$; a generator of the conjugate subgroup is obtained by replacing 1 by 2, 2 by 5 and 5 by 1 in a generator of H.)

Solution to Additional Exercise B51

Only the symbols 1, 4 and 5 appear in the list of elements of H; the symbols 2 and 3 are fixed. Similarly, only the symbols 2, 3 and 5 appear in the list of elements of K; the symbols 1 and 4 are fixed.

Consider the elements of K written down underneath those of H as in the question:

$$H = \{e,\ (1\ 4),\ (1\ 5),\ (4\ 5),\ (1\ 4\ 5),\ (1\ 5\ 4)\},$$

$$K = \{e,\ (2\ 3),\ (2\ 5),\ (3\ 5),\ (2\ 3\ 5),\ (2\ 5\ 3)\}.$$

Here the symbol 2 always appears below the symbol 1, the symbol 3 always appears below the

symbol 4, and the symbol 5 always appears below the symbol 5. So we can obtain a permutation f such that $K = f \circ H \circ f^{-1}$ by treating the lists of elements of H and K above as a 'two-line form' in the usual way, and also mapping the fixed symbols of H to the fixed symbols of K:

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ f & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 2 & 1 & 4 & 3 & 5. \end{array}$$

Thus a suitable permutation f is

$$f = (1\ 2)(3\ 4).$$

A permutation g such that $H = g \circ K \circ g^{-1}$ is

$$g = f^{-1} = f = (1\ 2)(3\ 4).$$

(There are other suitable conjugating permutations f and g here. For example, we could map the fixed symbols of H to the fixed symbols of K in the other order:

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ f & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 2 & 4 & 1 & 3 & 5. \end{array}$$

This gives

$$f = (1\ 2\ 4\ 3) \quad \text{and} \quad g = f^{-1} = (1\ 3\ 4\ 2).$$

We could also write the elements of K in a different way, making sure that we do not lose the property that in the resulting 'two-line form' each symbol in the second line always appears below the *same* symbol in the first line (for instance, we cannot have an occurrence of 2 appearing below 1 and another occurrence of 2 appearing below 4).

For example, we can write

$$H = \{e,\ (1\ 4),\ (1\ 5),\ (4\ 5),\ (1\ 4\ 5),\ (1\ 5\ 4)\},$$
$$K = \{e,\ (3\ 2),\ (3\ 5),\ (2\ 5),\ (3\ 2\ 5),\ (3\ 5\ 2)\}.$$

Then if we map the fixed symbols 2 and 3 of H to the fixed symbols 1 and 4 of K, respectively, we obtain

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ f & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 3 & 1 & 4 & 2 & 5, \end{array}$$

which gives

$$f = (1\ 3)(2\ 4) \quad \text{and} \quad g = f^{-1} = (1\ 3)(2\ 4).)$$

(In fact H is the subset of S_5 consisting of all the permutations in S_5 that fix both of the symbols 2 and 3, and similarly K is the subset of S_5 consisting of all the permutations in S_5 that fix both of the symbols 1 and 4. In general, if A is any subset of the set of symbols $\{1, 2, 3, \ldots, n\}$, then the permutations in S_n that fix all the symbols in A form a subgroup of S_n. One way to see this is to observe that the Cayley table for these permutations will look exactly the same as the group table for the group of all permutations of the set of symbols $\{1, 2, \ldots, n\} - A$. For example, the Cayley table for the permutations in S_5 that fix the symbols 2 and 3 will look exactly the same as the group table for the group of all permutations of the set of symbols $\{1, 4, 5\}$.)

Solution to Additional Exercise B52

(a) An element of S_5 of order 6 is $(1\ 2)(3\ 4\ 5)$, so a cyclic subgroup of S_5 of order 6 is

$$\langle (1\ 2)(3\ 4\ 5) \rangle = \{e,\ (1\ 2)(3\ 4\ 5),\ (3\ 5\ 4),\ (1\ 2),\ (3\ 4\ 5),\ (1\ 2)(3\ 5\ 4)\}.$$

(b) Each cyclic subgroup of order 6 is isomorphic to the subgroup in part (a) and therefore contains exactly two permutations of order 6. Since S_5 contains 20 permutations of order 6, it has $20/2 = 10$ cyclic subgroups of order 6.

(An element g of order k in a group G cannot occur in more than one subgroup of G of order k, because if $g \in K$ where K is a subgroup of G of order k then $K = \langle g \rangle$.)

Solution to Additional Exercise B53

Subgroups of S_5 of order 10 can be found by labelling the locations of the vertices of a regular pentagon with the symbols 1, 2, 3, 4 and 5, and writing the symmetries of the pentagon as permutations of these symbols. Different labellings give possibly different, but isomorphic (and conjugate), subgroups.

Labelling the vertex locations of the pentagon with 1, 2, 3, 4 and 5 gives the following subgroup of S_5:

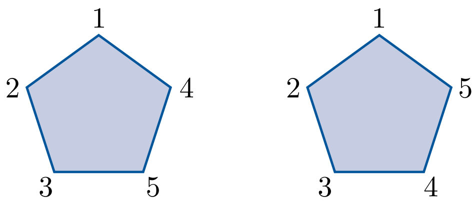
$$\begin{aligned}\{&e,\ (1\ 2\ 3\ 4\ 5),\ (1\ 3\ 5\ 2\ 4),\\ &(1\ 4\ 2\ 5\ 3),\ (1\ 5\ 4\ 3\ 2),\\ &(1\ 2)(3\ 5),\ (1\ 3)(4\ 5),\ (1\ 4)(2\ 3),\\ &(1\ 5)(2\ 4),\ (2\ 5)(3\ 4)\}.\end{aligned}$$

We can obtain another isomorphic subgroup by interchanging the symbols 4 and 5, for example, as

shown below. This gives:

$$\{e,\ (1\ 2\ 3\ 5\ 4),\ (1\ 3\ 4\ 2\ 5),\ (1\ 5\ 2\ 4\ 3),\ (1\ 4\ 5\ 3\ 2),\ (1\ 2)(3\ 4),\ (1\ 3)(4\ 5),\ (1\ 5)(2\ 3),\ (1\ 4)(2\ 5),\ (2\ 4)(3\ 5)\}.$$

(Twelve different subgroups of S_5 of order 10 can be found by labelling the pentagon in different ways.)



Solution to Additional Exercise B54

Using the method of Worked Exercise B47, we obtain the following permutations:

$\circ$	e	a	p	x	q	y	r	z		
e	e	a	p	x	q	y	r	z	$\longrightarrow$	i (identity)
a	a	e	x	p	y	q	z	r	$\longrightarrow$	$(e\ a)(p\ x)(q\ y)(r\ z)$
p	p	x	a	e	r	z	y	q	$\longrightarrow$	$(e\ p\ a\ x)(q\ r\ y\ z)$
x	x	p	e	a	z	r	q	y	$\longrightarrow$	$(e\ x\ a\ p)(q\ z\ y\ r)$
q	q	y	z	r	a	e	p	x	$\longrightarrow$	$(e\ q\ a\ y)(p\ z\ x\ r)$
y	y	q	r	z	e	a	x	p	$\longrightarrow$	$(e\ y\ a\ q)(p\ r\ x\ z)$
r	r	z	q	y	x	p	a	e	$\longrightarrow$	$(e\ r\ a\ z)(p\ q\ x\ y)$
z	z	r	y	q	p	x	e	a	$\longrightarrow$	$(e\ z\ a\ r)(p\ y\ x\ q)$

Hence a permutation group isomorphic to the given group is

$$\{i,\ (e\ a)(p\ x)(q\ y)(r\ z),\ (e\ p\ a\ x)(q\ r\ y\ z),\ (e\ x\ a\ p)(q\ z\ y\ r),\ (e\ q\ a\ y)(p\ z\ x\ r),\ (e\ y\ a\ q)(p\ r\ x\ z),\ (e\ r\ a\ z)(p\ q\ x\ y),\ (e\ z\ a\ r)(p\ y\ x\ q)\},$$

where i is the identity permutation.

Additional exercises for Unit B4

Section 1

Additional Exercise B55

(a) For each of the orders of a group G given below, write down all the possible orders of subgroups of G.

(i) 17 (ii) 36 (iii) 56 (iv) 59

(b) In which of the cases above is G necessarily a cyclic group?

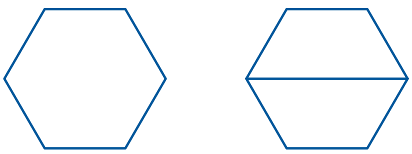
(c) In which of the cases above might G be a cyclic group?

(d) Determine the order of the subgroup of the group $(\mathbb{Z}_{59}, +_{59})$ generated by the element 7.

Additional Exercise B56

The group $S(\hexagon)$, the symmetry group of the regular hexagon, has order 12, so by Lagrange's Theorem the possible orders of its subgroups are 1, 2, 3, 4, 6 and 12. It has a subgroup of order 4, because if we modify the hexagon by adding a line segment joining two opposite vertices as shown on the right below, then the symmetries of the modified figure form a subgroup of $S(\hexagon)$ whose elements are the following four symmetries:

- the identity symmetry
- rotation through π
- reflection in the line of which the new line segment is a part
- reflection in the line through the centre perpendicular to the new line segment.



Modify the hexagon in four other ways to obtain four modified figures whose symmetries form subgroups of $S(\hexagon)$ of orders 1, 2, 3 and 6, respectively, and describe the symmetries of the modified figure in each case.

Section 2

Additional Exercise B57

The Cayley tables below define groups. Let these groups be $(P, \circ)$ and $(Q, \circ)$, respectively.

$\circ$	e	a	b	c	p	q	r	s
e	e	a	b	c	p	q	r	s
a	a	b	c	e	q	r	s	p
b	b	c	e	a	r	s	p	q
c	c	e	a	b	s	p	q	r
p	p	q	r	s	e	a	b	c
q	q	r	s	p	a	b	c	e
r	r	s	p	q	b	c	e	a
s	s	p	q	r	c	e	a	b

$(P, \circ)$

$\circ$	e	i	j	k	w	x	y	z
e	e	i	j	k	w	x	y	z
i	i	w	k	y	x	e	z	j
j	j	z	w	i	y	k	e	x
k	k	j	x	w	z	y	i	e
w	w	x	y	z	e	i	j	k
x	x	e	z	j	i	w	k	y
y	y	k	e	x	j	z	w	i
z	z	y	i	e	k	j	x	w

$(Q, \circ)$

(a) State a standard group that is isomorphic to $(P, \circ)$.

(b) State a standard group that is isomorphic to $(Q, \circ)$.

(c) Show that $\{e, i, w, x\}$ is a subgroup of $(Q, \circ)$, and state a standard group that is isomorphic to this subgroup.

Additional Exercise B58

Find a subgroup of the group $(P, \circ)$ in Additional Exercise B57 that is isomorphic to the Klein four-group V, justifying your answer.

Additional Exercise B59 Challenging

You briefly met the quaternion group

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

in Subsection 2.5 of Unit B4. The structure of this group, including the Cayley table given in Unit B4, can be deduced by assuming that the set

$$\{1, -1, i, -i, j, -j, k, -k\}$$

is a group and that the following facts hold.

- 1 is the identity.
- -1 commutes with every other element.
- $(-1)^2 = 1$.
- $(-1)i = -i$, $(-1)j = -j$ and $(-1)k = -k$.
- $i^2 = j^2 = k^2 = ijk = -1$.

For example, $(-i)^2 = -1$ can be deduced as follows:

$$\begin{aligned}(-i)(-i) &= ((-1)i)((-1)i)\\ &= (-1)^2(i)^2\\ &= 1(-1) = -1,\end{aligned}$$

since multiplication is associative.

(Note that each of the facts above is used in this deduction.)

Answer the following in a similar manner; that is, using these facts and without reference to the Cayley table for Q_8 given in Unit B4.

(a) Show that i and $-i$ are inverses of each other.

(b) Show that i has order 4.

(c) Find $\langle j\rangle$ and $\langle k\rangle$, and hence write down j^{-1} and k^{-1}.

(d) Find $(-1)(-i)$, ij, jk and ik.

(e) Show that Q_8 is non-abelian.

Section 3

Additional Exercise B60

The following theorem is from Subsection 2.1 of Unit B4.

Theorem B74

Let G be a group of order greater than 2 in which each element except the identity has order 2. Then the order of G is a multiple of 4.

(a) Rewrite the theorem in the form 'If ..., then ...'.

(b) Identify the hypothesis (or hypotheses) and conclusion (or conclusions) of the theorem.

(c) Which of the following are correct versions of this theorem?

  (i) If G is a group in which each element except the identity has order 2, then the order of G is a multiple of 4 provided that the order of G is greater than 2.

  (ii) Let G be a group of order greater than 2. Then each element except the identity has order 2 provided that the order of G is a multiple of 4.

  (iii) Let G be a group of order greater than 2. If each element of G except the identity has order 2, then the order of G is a multiple of 4.

Additional Exercise B61

Let G be a group, and let x be an element of G.

Assuming only the group axioms and Propositions B11 and B12, prove that if g is an element of G such that $xg = x$, then $g = e$.

Additional Exercise B62

Let a, b and c be elements of a group G.

Prove that there exists a unique element x in G such that $axb = c$.

(Note that both existence and uniqueness need to be proved.)

Additional Exercise B63

Let G be an abelian group.

Prove that the subset $H = \{g \in G : g^2 = e\}$ is a subgroup of G.

Additional Exercise B64

Let G be a group, and let H, K and L be subgroups of G.

Prove that $H \cap K \cap L$ is a subgroup of G.

Additional Exercise B65

Let H and K be subgroups of a group G.

Prove that $H - K$ is not a subgroup of G.

Additional Exercise B66 Challenging

Let H and K be subgroups of a group G.

(a) Prove that if $h \in H - K$ and $k \in K - H$ then $hk \notin H \cup K$.

(b) Hence prove that $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.

Additional Exercise B67 Challenging

Let S be a subset of a finite group G.

Prove that S is a subgroup of G if and only if S is non-empty and $a, b \in S$ implies that $ab \in S$.

Solutions to additional exercises for Unit B4

Solution to Additional Exercise B55

(a) It follows from Lagrange's Theorem that the possible orders of the subgroups of a group G are the positive divisors of $|G|$.

(i) The possible orders are 1 and 17.

(ii) The possible orders are 1, 2, 3, 4, 6, 9, 12, 18 and 36.

(iii) The possible orders are 1, 2, 4, 7, 8, 14, 28 and 56.

(iv) The possible orders are 1 and 59.

(b) Groups of prime order are necessarily cyclic by Corollary B70 to Lagrange's Theorem, so the groups of order 17 and 59 are cyclic.

(c) The group G might be cyclic in all cases. For each positive integer n there is a cyclic group of order n, namely $(\mathbb{Z}_n, +_n)$.

(In fact there do exist non-cyclic groups of orders 36 and 56.)

(d) The group $(\mathbb{Z}_{59}, +_{59})$ is a cyclic group of prime order. By Corollary B70 to Lagrange's Theorem, each element, except the identity, generates the whole group. So, in particular, 7 is a generator of $(\mathbb{Z}_{59}, +_{59})$; that is, $\langle 7\rangle = \mathbb{Z}_{59}$. Thus 7 generates a subgroup of order 59.

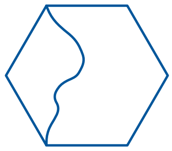
(Alternatively, we could use Corollary B69 to Lagrange's Theorem. By this corollary, the order of the element 7 of $(\mathbb{Z}_{59}, +_{59})$ divides the order of $(\mathbb{Z}_{59}, +_{59})$, which is 59. Hence, since 59 is prime and 7 is not the identity element, the order of 7 must be 59. Hence 7 generates a subgroup of order 59; that is, it generates the whole group.)

(Alternatively again, but less easily, we can work this out using results about the group $(\mathbb{Z}_n, +_n)$ from Subsection 3.4 of Unit B2. For example, Theorem B38 tells us that a non-zero element m of the group $(\mathbb{Z}_n, +_n)$ has order n/d, where d is the highest common factor of m and n. Since 59 is prime, the highest common factor of 7 and 59 is 1, and hence the order of 7 in the group $(\mathbb{Z}_{59}, +_{59})$ is $59/1 = 59$. Thus the order of the subgroup $\langle 7\rangle$ is also 59.)

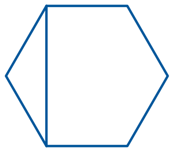
Solution to Additional Exercise B56

There are different possibilities for the modified figures, but possible answers are given here.

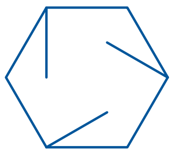
The modified figure below has a symmetry group of order 1. Its only element is the identity symmetry.



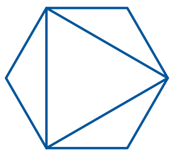
The modified figure below has a symmetry group of order 2. Its elements are the identity symmetry and the reflection in the horizontal line through the centre of the hexagon.



The modified figure below has a symmetry group of order 3. Its elements are the identity symmetry and the rotations through $2\pi/3$ and $4\pi/3$ about the centre of the hexagon.



The modified figure below has a symmetry group of order 6. Its elements are the identity symmetry, the rotations through $2\pi/3$ and $4\pi/3$ about the centre of the hexagon, and the three reflections in the axes of symmetry of the central triangle.



Solution to Additional Exercise B57

(a) The group $(P, \circ)$ is abelian and it has three elements of order 2, so it is isomorphic to $(U_{15}, \times_{15})$ (isomorphism class 3 for groups of order 8).

(b) The group $(Q, \circ)$ is non-abelian and it has only one element of order 2, so it is isomorphic to the quaternion group Q_8 (isomorphism class 5 for groups of order 8).

(c) We show that the three subgroup properties hold for $(\{e, i, w, x\}, \circ)$. The Cayley table for $\{e, i, w, x\}$ is as follows.

$\circ$	e	i	w	x
e	e	i	w	x
i	i	w	x	e
w	w	x	e	i
x	x	e	i	w

SG1 Every element in the table is in $\{e, i, w, x\}$, so this set is closed under $\circ$.

SG2 The identity in $(Q, \circ)$ is e, and $e \in \{e, i, w, x\}$.

SG3 From the Cayley table, we see that the inverse of each element of $\{e, i, w, x\}$ is in $\{e, i, w, x\}$, as below.

Element	e	i	w	x
Inverse	e	x	w	i

Hence $\{e, i, w, x\}$ satisfies the three subgroup properties, and so is a subgroup of $(Q, \circ)$.

This subgroup has order 4 and contains only two self-inverse elements, so it is isomorphic to the cyclic group C_4.

Solution to Additional Exercise B58

The group V contains three elements of order 2. The elements of order 2 in $(P, \circ)$ are b, p and r. We show that $\{e, b, p, r\}$ is a subgroup of $(P, \circ)$ isomorphic to V.

First we show that the three subgroup properties hold for $(\{e, b, p, r\}, \circ)$. The Cayley table for $\{e, b, p, r\}$ is as follows.

$\circ$	e	b	p	r
e	e	b	p	r
b	b	e	r	p
p	p	r	e	b
r	r	p	b	e

SG1 Every element in the table is in $\{e, b, p, r\}$, so this set is closed under $\circ$.

SG2 The identity in $(P, \circ)$ is e, and $e \in \{e, b, p, r\}$.

SG3 From the Cayley table, we see that each element of $\{e, b, p, r\}$ is self-inverse and hence its inverse is in $\{e, b, p, r\}$.

Hence $\{e, b, p, r\}$ satisfies the three subgroup properties, and so is a subgroup of $(P, \circ)$.

This subgroup is of order 4 and all of its elements are self-inverse, so it is isomorphic to the Klein four-group V.

Solution to Additional Exercise B59

(a) We have

$$(-i)i = ((-1)i)i = (-1)i^2 = (-1)^2 = 1,$$

since $(-1)i = -i$, $i^2 = -1$ and $(-1)^2 = 1$ and multiplication is associative.

Multiplying both sides of the equation $(-i)i = 1$ on the right by i^{-1} gives $-i = i^{-1}$. Hence i and $-i$ are inverses of each other.

(b) We have $i^2 = -1$ and so

$$i^3 = (-1)i = -i,$$

and

$$i^4 = (-i)i = 1 \quad \text{(by part (a))}.$$

(Alternatively, $i^4 = (i^2)^2 = (-1)^2 = 1$.)

Therefore 4 is the smallest natural number n such that $i^n = 1$, and hence i has order 4.

(c) We have $j^2 = -1$, so

$$j^3 = (-1)j = -j$$

and

$$j^4 = (j^2)^2 = (-1)^2 = 1.$$

Therefore

$$\langle j \rangle = \{1, j, -1, -j\}.$$

We have $j^4 = 1$, so $j^{-1} = j^3 = -j$.

Similar reasoning gives

$$\langle k \rangle = \{1, k, -1, -k\},$$

and $k^{-1} = -k$.

(d)
$$\begin{aligned} (-1)(-i) &= (-1)((-1)i) = (-1)^2 i = i \\ ij &= (ijk)k^{-1} = (-1)(-k) = k \\ jk &= i^{-1}(ijk) = (-i)(-1) = i \\ ik &= i(ij) = i^2 j = (-1)j = -j. \end{aligned}$$

(e) We have $ij = k$ (by part (d)), but

$$\begin{aligned} ji &= j(jk) \quad \text{(by part (d))} \\ &= j^2 k = (-1)k = -k. \end{aligned}$$

Therefore $ij \neq ji$ and hence Q_8 is non-abelian.

(Some other pairs of elements do not commute: for example j and k do not commute because $jk = i$ and $kj = (ij)j = ij^2 = i(-1) = -i$.)

Solution to Additional Exercise B60

(a) The theorem can be rewritten as

If G is a group of order greater than 2 in which each element except the identity has order 2, then the order of G is a multiple of 4.

(b) The hypotheses are:

- G is a group of order greater than 2
- each element of G except the identity has order 2.

The conclusion is:

- the order of G is a multiple of 4.

Statements (i) and (iii) are correct versions of the theorem, and statement (ii) is not.

Solution to Additional Exercise B61

Let g be an element of G such that

$$xg = x.$$

Composing each side of this equation by x^{-1} on the left gives

$$x^{-1}xg = x^{-1}x.$$

Therefore

$$eg = x^{-1}x \quad \text{(by axiom G4, inverses)},$$

so

$$g = e \quad \text{(by axiom G3, identity)},$$

as required.

Solution to Additional Exercise B62

Let $x = a^{-1}cb^{-1}$. Then

$$\begin{aligned} axb &= aa^{-1}cb^{-1}b \\ &= ece \quad \text{(by axiom G4, inverses)} \\ &= c \quad \text{(by axiom G3, identity).} \end{aligned}$$

This confirms the existence of a suitable element x.

Now we prove that this element x is unique. Suppose that x and y are elements of G such that $axb = c$ and $ayb = c$. Then

$$axb = ayb,$$

and, by the Cancellation Laws,

$$x = y.$$

This confirms that the element x is unique.

Thus there exists a unique element x such that $axb = c$, as required.

(Another way to show uniqueness is as follows. This method is also a way of finding the expression $a^{-1}cb^{-1}$ used above.

Suppose that x is an element of G such that

$$axb = c.$$

Composing on the left with a^{-1} and on the right with b^{-1} gives

$$a^{-1}axbb^{-1} = a^{-1}cb^{-1};$$

that is,

$$exe = a^{-1}cb^{-1} \quad \text{(by axiom G4, inverses)},$$

which simplifies to

$$x = a^{-1}cb^{-1} \quad \text{(by axiom G3, identity).}$$

This shows that the only possibility for an element x satisfying the equation $axb = c$ is the element $x = a^{-1}cb^{-1}$.)

Solution to Additional Exercise B63

The set $H = \{g \in G : g^2 = e\}$ is a subset of G. We show that it satisfies the three subgroup properties.

SG1 Let $g, h \in H$. Then $g^2 = e$ and $h^2 = e$. To show that $gh \in H$, we show that $(gh)^2 = e$. Now

$$\begin{aligned} (gh)^2 &= ghgh \\ &= gghh \quad \text{(since } G \text{ is abelian)} \\ &= g^2h^2 \\ &= e. \end{aligned}$$

Thus H is closed (under the binary operation of G).

SG2 We have $e^2 = e$, so $e \in H$.

SG3 Let $g \in H$. Then $g^2 = e$. This equation implies that $g^{-1} = g$, so $g^{-1} \in H$. Thus H contains the inverse of each of its elements.

Hence H satisfies the three subgroup properties, so it is a subgroup of G.

(An alternative way to check property SG3, a little less neat than the way above, is as follows.

SG3 Let $g \in H$. Then $g^2 = e$. To show that $g^{-1} \in H$, we show that $(g^{-1})^2 = e$. Now

$$\begin{aligned} (g^{-1})^2 &= (g^2)^{-1} \quad \text{(by an index law)} \\ &= e^{-1} \\ &= e. \end{aligned}$$

Thus H contains the inverse of each of its elements.)

Solution to Additional Exercise B64

We could prove that $H \cap K \cap L$ is a subgroup of G by checking the three subgroup properties.

However, an alternative way to prove it is to use the fact that the intersection of any *two* subgroups of a group is a subgroup, by Theorem B81. The proof is as follows.

Since H and K are subgroups of G, it follows from Theorem B81 that $H \cap K$ is a subgroup of G. Hence, since L is also a subgroup of G, it follows from the same result that $(H \cap K) \cap L = H \cap K \cap L$ is a subgroup of G, as required.

Solution to Additional Exercise B65

The identity element e of G lies in both H and K, so $e \notin H - K$. Thus subgroup property SG2 (identity) fails for $H - K$. Hence $H - K$ is not a subgroup of G.

(Note that we have to prove the result in the question for *all* groups G and *all* subgroups H and K of G, so we need to give a general proof. A specific example where $H - K$ is not a subgroup of G is not enough.)

Solution to Additional Exercise B66

(a) Suppose that $h \in H - K$ and $k \in K - H$. Then $hk \notin H$, because if $hk \in H$ then, since H is a subgroup, we would have $h^{-1}hk \in H$; that is, $k \in H$. Similarly $hk \notin K$, because if $hk \in K$ then, since K is a subgroup we would have $hkk^{-1} \in K$; that is, $h \in K$. Since $hk \notin H$ and $hk \notin K$, we have $hk \notin H \cup K$.

(b) 'If' part Suppose that $H \subseteq K$ or $K \subseteq H$. Then $H \cup K = K$ or $H \cup K = H$. Both H and K are subgroups of G, so it follows that $H \cup K$ is a subgroup of G.

'Only if' part We have to prove that

> If $H \cup K$ is a subgroup of G, then $H \subseteq K$ or $K \subseteq H$.

We prove the contrapositive, which is

> If $H \nsubseteq K$ and $K \nsubseteq H$, then $H \cup K$ is not a subgroup of G.

Suppose that $H \nsubseteq K$ and $K \nsubseteq H$. Then there is an element h in $H - K$ and an element k in $K - H$. Both h and k are elements of $H \cup K$, but, by part (a), $hk \notin H \cup K$. Thus subgroup property SG1 (closure) fails for $H \cup K$. Hence $H \cup K$ is not a subgroup of G, as required.

This completes the proof.

Solution to Additional Exercise B67

'Only if' part Suppose that S is a subgroup of G. We have to show that S is non-empty and that if $a, b \in S$ then $ab \in S$.

Since S is a subgroup, $e \in S$ by subgroup property SG2, so S is non-empty.

Also since S is a subgroup, S is closed by subgroup property SG1; that is, if $a, b \in S$ then $ab \in S$.

'If' part Suppose that S is a non-empty subset of G, and that if $a, b \in S$ then $ab \in S$. We show that the three subgroup properties hold for S.

SG1 By our supposition, if $a, b \in S$ then $ab \in S$; that is, S is closed under the binary operation of G.

SG2 Since S is non-empty, there is an element a in S. Since the composite of any two elements in S is also in S, it follows that $a^2 \in S$, and hence that $a^3 = a^2 a \in S$, and so on. That is, all of

$$a, a^2, a^3, \ldots$$

are in S. Since a is an element of the finite group G, it has finite order, say n. Hence

$$a^n = e,$$

and therefore $e \in S$.

SG3 Let $a \in S$. Then, as shown above, all of

$$a, a^2, a^3, \ldots$$

are in S, and a has finite order, say n.

If $n = 1$, then $a = e$, so $a^{-1} \in S$.

Otherwise, $a^{-1} = a^{n-1}$, since

$$aa^{n-1} = a^{n-1}a = a^n = e,$$

so, again, $a^{-1} \in S$.

Thus S contains the inverse of each of its elements.

Hence S satisfies the three subgroup properties, and hence it is a subgroup of G. This completes the proof.